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# Teaching Vibration to university undergraduates

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**Abstract**
This paper provides a description of an undergraduate course on vibration, given to second and third year students at Bristol University in the UK. The course, and my teaching philosophy, were developed over more than 30 years. The lectures were given in two 20-hour courses which were supported by an equal number of examples classes. Students were provided with a series of question sheets which contained questions from previous examination papers. In addition, laboratory classes were provided to give the students "hands on" experience on how to excite, control, measure, and to interpret various vibrating systems. The first set of lectures began with the analysis of a single degree of freedom system, adding different forcing functions and more degrees of freedom. The second set of lectures introduced continuous systems, consisting of bars, beams, and plates. The limitations of reality, particularly boundary conditions, was emphasized. Wherever possible, some artifact was taken to the lecture amplify the mathematics. I have interleaved into the presentation some of my teaching philosophy and how it is important in a heavily mathematical subject such as vibration to teach rather than to try and impress the students as to how clever is their lecturer. Finally, if the lecturer does not enjoy giving the lectures, the students will not receive that "extra" which distinguishes a good lecture from a bad lecture, and also distinguishes a good lecture from reading a text book.

## 1. Philosophy of teaching

As this is a personal account, I ask you to forgive me for using *I, me* instead of a more formal mode of writing. I gave a series of lectures on Vibration in the second and third years of the Mechanical Engineering course at the University of Bristol, each course having about 20 lectures and supported by examples classes and laboratories.

Many books have been written on vibrations, each using their own style and emphasis. Unfortunately, I did not find a single book which I could recommend to my students to support my lecture courses. Books are often written to impress publishing committees and their advisors, so must be different from what has been done before. As a professor, I could quickly identify books which were written to impress professors rather than to be useful and intelligible to students.

Engineering [and other] courses change with time; some material will be dropped while other material will be added. A good course will contain a spectrum of subject matter so that a student will graduate with a knowledge and understanding which should be not only immediately useful in his/her new job but will sustain the necessary knowledge acquisition tools during his/her career. The first question to be asked about the teaching of any part of any subject is "Why should we teach this"? Questions as to when, how, and how deep will surely follow. Some subjects, such as electronics and computer science have moved very quickly in the last few years while subjects such as dynamics, elasticity, plasticity, and thermodynamics have hardly changed, while they are still necessary to understand modern engineering. But some people have odd perspectives. I was once asked by a lecturer in electronics why we still taught steam in our thermodynamics courses. He had no concept as to how electricity was generated via steam turbines and to gain a tiny increase in efficiency had major economic implications. Vibrations, being a major subset of dynamics, is important in all aspects of engineering where structural forms can be subject to some form of excitation. It is therefore necessary to understand how structures respond and what is the nature of the excitation. These days, complex structures can be analysed using computer programs such as finite element [FE] analysis. However, to understand the limitations and pitfalls of these programs, the basic building blocks on which they are based must be understood. This also includes efficient approximate methods, such as Rayleigh and Stodola, for obtaining solutions.

In vibrations, real systems are modelled so that we can understand why a structure or system responds in the way it does to some sort of excitation. We teach simple [lumped] systems where the mass, stiffness and damping are separate, and systems such as bars, beams, and plates where the mass and stiffness are "distributed". In my own teaching, the lumped systems were in year 2 and the distributed systems in year 3. Both lumped and distributed systems can quickly become very complicated mathematically, except for very simple cases. Without a thorough understanding of those simple systems, FE and similar models cannot be created and only the unwise use them without a good knowledge of the underlying mechanics.

There is an unfortunate tendency in universities to teach only that which can be examined. It leads to lots of mathematical manipulations which are readily marked in examinations. However, the useful part of engineering is not just solving algebra but understanding the limits of such algebra we have applied to a real engineering situation. Unfortunately, this "useful" material is not easily taught and examined as it requires a maturity of experience which undergraduates do not have. However, the lecturer ought to have some relevant experience from his/her research, experience in industry, or consulting. Imparting experience forms a useful means of breaking up a solid hour of mathematics and allows some of the slower members of the class to catch up. All classes will have a range of ability and my experience of teaching means it is not easy to keep everyone happy all the time. Some of the "whiz kids" can get impatient with deviations from straight line mathematics as all they are interested in is doing well in the examinations; such people do not always become good engineers. At a dinner, I asked a student from another engineering Department which lecturer was generally thought of [by the students] as the cleverest in his Department. His answer was Dr X. This surprised me as Dr X had a very light research record, so I asked why he was the cleverest. He said that Dr X's lectures were so difficult to follow that he must be very clever as only he seemed to be able to understand them. In contrast, there is also the danger of making lectures too easy to understand so students lose concentration and hence the thread of the lecture. For this reason, it is vital to be able to see the faces of the students as this will give a good indication as to whether they are bored or baffled. It is also important to encourage students to ask questions during a lecture since for every bold student, there will be ten timid ones who would have liked to have asked that question but did not have the courage to do so. Above all, with a subject such as vibrations, where there is a substantial mathematical content, the lecturer must communicate and not just try to look clever.

I was brought up on "chalk and talk" whereas the current trend is to use Powerpoint or similar presentation methods. With Powerpoint, the students can read the equations easily and many institutions arrange for electronic copies of the lecture to be available. But if the students are to understand what is being presented, they must follow the steps in any argument being developed. It is not for nothing that the main lecture theatre in the newly built Maths Department at Oxford is equipped with a battery of whiteboards [although they do also have the facilities for more modern projection methods]. I am old fashioned enough to believe in teaching rather than lecturing. A lecture presentation is an ego trip, while teaching is the successful communication of a message. For that reason, I never presented from notes [even though I had them with me] but developed my argument directly from my head. The basis for this way of teaching was that students ought to be able to follow me in real time as the rate of my presentation should, approximately, match their rate of understanding. When using Powerpoint or similar, it is not easy to pace the presentation, particularly with respect to laying out line by line steps. Of course, there are those who will call me old fashioned, to which I say OK, you do it your way and I will do it mine. ***In over 30 years of teaching vibration, I have learned what works and what does not.***

Some lecturers are lazy when setting questions in exams. Lazy questions are of the form "derive from first principles…" or "show that…", and are simply tests of mathematical skill which are easy to mark. Being mathematical, Vibration has many opportunities for lazy questions, but these do not draw out any understanding of the subject. In both my courses, I gave out a crib sheet containing all the basic equations [and their solutions] early in the course and told the students that an identical crib sheet would be given out in the exams. These crib sheets [for year 2 and years 3] are given in the supplementary information to this paper [SI_Crib sheets_2_and_3]. I invited the students to advise me if they wanted anything to be added, but nothing ever was. The objective was that the students were not expected to derive standard equations, but to use the given equations to solve questions posed as stories. Many enlightened universities provide students with quite extensive help books covering a wide variety of courses. For example, the Department of Engineering Science at Oxford University provides an extensive 180 page book authored by Howatson, Lund, and Todd. The book is issued to every student and is given out with the question paper in all exams.

## 2. Cautions

Students do not like torsional vibration. They find the sign conventions much more challenging than for linear problems. Curiously, this applied more to lumped parameter models than for distributed systems such as bars. Extra care must therefore be taken in teaching torsional motion and strict sign conventions used.

I found it easier to use the Heaviside Operator ***D*** *[=iω]* rather than Laplace Transforms when deriving solutions. The Laplace Transform gives both a steady state [Particular Integral] and a transient [Complementary Function] solution when all I needed was the steady state response to a forcing function.

Experience showed that students preferred a solution in the form of $cos[\omega t-\varphi]$ where $\varphi$ is a phase angle rather than using the complex form such as $e^{i\omega t}$. Somehow, it seemed to be more tangible to use real values rather than imaginary numbers.

I was once persuaded to use the concept of Receptances for teaching more complex systems, but this led to groans of anguish and I quickly reverted to more conventional equations. Receptances were used prior to the advent of digital computers and were a useful tool in their day, but that day has long passed.

## 3. What do we teach and how to get good outcomes

All universities and colleges will have different resources and will allocate different weighting to their menu of courses. I will describe what I did at Bristol University when I taught there on Vibrations for a period of over 30 years. I gave 20 lectures, each of 50 minutes duration, in years 2 and 3. While I am not trying to write another book, it will be necessary to describe in some detail what was taught.

### *3.1. Teaching programme*

In Year 2, an introductory lecture was used to outline the course and to talk generally about why engineers need to understand what happens due to unwanted vibration. Problems of fatigue, noise, discomfort, and component malfunction were described. The mathematical representation of vibration was introduced using a vector, $u_0$, rotating at an angular frequency $\omega$ such that the projection on a horizontal plane is $u_0 \cos \omega t$ where $t$ is the time from some datum as shown in Figure 1.

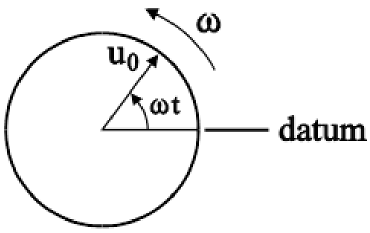



**Figure 1**: Mathematical representation of harmonic vibration

The angular frequency $\omega$ *[rad/sec]* $[=2\pi F]$ is used for convenience; $F$ is the frequency and has units 1/time or, scientifically, Hertz [Hz] and is such that $\omega = 2\pi F$. It is easily shown that for a vector rotating at an angular velocity $\omega$, the period of one oscillation is $\tau$ such that $\omega\tau = 2\pi$ and that $F\tau = 1$. The variation of amplitude with time is given in Figure 2.

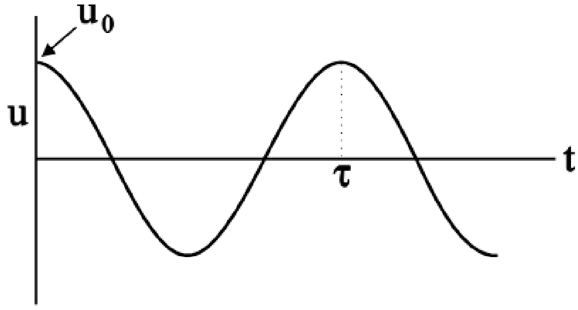



**Figure 2**: Variation of amplitude with time

This oscillatory form of $u(t)$ is said to be characterised in the *time domain*. We can also show it in the *frequency domain*, as in Figure 3.

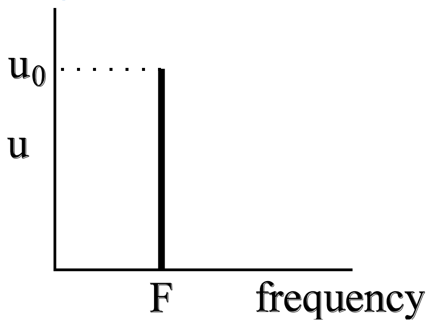



**Figure 3**: Frequency domain representation

### *3.2. Free vibration, Single degree of freedom system [1DoF]*

The first few lectures considered a single degree of freedom system consisting of a mass, spring, and a viscous damper, as shown in Figure 4.

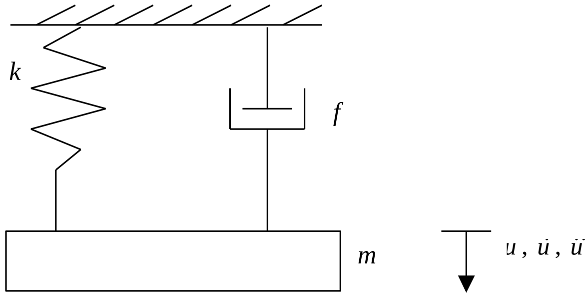



**Figure 4**: Single degree of freedom system consisting of a mass, spring, and dashpot

By considering the forces acting on the mass, the basic equation:

$$m\ddot{u} + f\dot{u} + ku = 0 \quad (1)$$

can be derived. In my notation, *u,v,* and *w* are displacements in the *x, y,* and *z* directions, *m* is the mass, *f* is the dashpot constant and *k* is the spring stiffness. The solution to Equation 1 is of the form:

$$u = Ue^{\alpha t}$$

such that

$$m\alpha^2 + f\alpha + k = 0$$

This equation has two roots given by:

$$\alpha_1, \alpha_2 = -\frac{f}{2m} \pm \sqrt{\frac{f^2}{4m^2} - \frac{k}{m}}$$

and can be re-written to give:

$$\frac{f}{2\sqrt{km}} = c, \quad \omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \alpha_1, \alpha_2 = \omega_n\left[-c \pm \sqrt{c^2 - 1}\right],$$

The concept of critical damping was introduced such that:

$$c = f / f_c$$

where $c$ is the proportion of critical damping and $f_c$ is the critical dashpot constant. It was explained that critical damping [$c$ = 1] defines the border between an oscillatory and a non-oscillatory response.

If $c > 1$, the response is non-oscillatory and is given by:

$$u = e^{-\omega_n ct}\left[Ae^{\omega_n t\sqrt{c^2-1}} + Be^{-\omega_n t\sqrt{c^2-1}}\right]$$

If $c < 1$, the response is oscillatory and is given by:

$$u = e^{-\omega_n ct}\left[C\cos\omega_n t\sqrt{1-c^2} + D\sin\omega_n t\sqrt{1-c^2}\right]$$

The special case of $c = 1$ has a response given by:

$$u = (A + Bt)e^{-\omega_n t}$$

Note that Equation 1 was developed for a horizontally moving mass where gravity is not considered. But if the mass is hanging vertically, the position it sits at rest, the equilibrium position, must be defined and oscillation defined from that datum. Otherwise, the equations become rather messy, and it is easy to make mistakes.

The students know what a mass and a spring look like, but few will have met a viscous dashpot. Having worked on my own cars, I had a damper or two in my garage, so these were handed round the class to get the "feel" of how they acted. As most students are interested in cars, they all know the terms "damper" and "shock absorber" and use them interchangeably. Now is the opportunity to illustrate the lectures on single degree of freedom systems without any excitation term. A damper serves to reduce any oscillation of the car body relative to the wheels. Until the mid 1930s, most cars used friction dampers. These consisted of friction discs compressed by a central bolt. If the bolt was too tight, the damper was locked solid and so did not work. If the bolt was too slack, there was little frictional resistance and so little damping. These friction dampers were replaced by oil filled telescopic viscous dampers which have a stronger force in one direction than the other for a given velocity. Why? This is where the shock absorber behaviour is revealed. A 1000kg car will have each wheel supporting about 250kg, so the spring will normally have a compressive load of 2500N. If the wheel hits a raised bump in the road, the wheel is forced upwards, compressing the spring and damper. The force in spring cannot be avoided [hard or soft springs…] but the force in the damper is proportional to the dashpot constant. To reduce the force transmitted to the vehicle body, this dashpot constant should ideally be zero, or as small as possible. Now consider the opposite case where the wheel encounters a downward depression or hole in the road. A typical wheel might have a combined mass [rim plus tyre] of 15kg for a 1000kg vehicle. We now have a force of 2500N acting on a 15kg mass. The spring will accelerate the wheel towards the bottom of the hole where it will suddenly stop, causing a severe impulse on the car body. This is where the shock absorber function comes into action. If the damper constant is high, the downwards acceleration of the wheel, and hence the impulse, will be reduced. This is why the dashpot constants in the two directions are different. Of course, for practical considerations, there has to be a compromise, and note that modern telescopic dampers can be much more complicated than a simple piston in a tube.

Equation 1 and its solution are very useful in illustrating not only the need to understand vibration, but also how this affects everyday events. Explaining to the students that the damping in a typical modern car is near to critical and this can be seen by pushing a car up and down and then seeing how long it takes to come to rest, is left to the discretion of the lecturer…

**It was also pointed out that viscous damping is a mathematical convenience.** Apart from certain polymers and oil based viscous dampers, material damping is hysteretic and not frequency dependent, but is often amplitude dependent. Also, much structural damping is due to friction which is certainly not viscous.

### *3.3. Forced vibration, 1 DoF*

Next comes forced vibration where there are two cases, neither of which is exactly followed in practice. In the first, a harmonic force of magnitude $P$ acts on the mass. This magnitude of this force is usually kept constant and it is assumed that the force can move through any distance as the mass moves, even at resonance. In this case, we have:

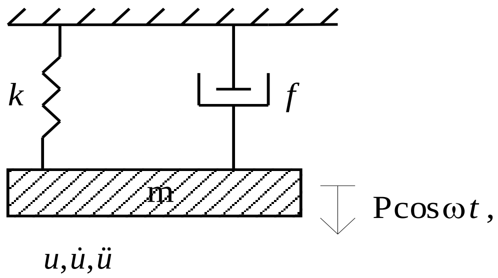



**Figure 5**: Single degree of freedom system with forced excitation

From the forces acting on the mass shown in Figure 5 and applying Newton's second law, the equation of motion is given by:

$$m\ddot{u} + f\dot{u} + ku = P\cos\omega t \tag{2}$$

The solution of which in the steady state is given by the Frequency Response Function, FRF:

$$u = \frac{P\cos(\omega t - \phi)}{k\sqrt{(1-r^2)^2 + (2cr)^2}}$$

Where:

$$\tan\phi = \frac{2cr}{1-r^2} \quad , \quad r = \frac{\omega}{\omega_n}$$

In practice, the force is often due to a rotating out of balance or a reciprocating component such that the magnitude of the force is proportional to $\omega^2$. This is particularly important at frequencies above resonance.

The second case in forced vibration is where there is abutment [earthquake] excitation as shown in Figure 6.

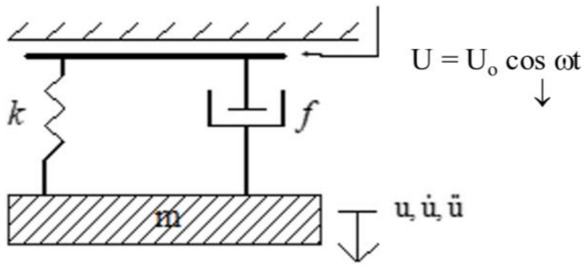



**Figure 6**: Single degree of freedom system with abutment excitation

Here, it is assumed that whatever is causing the excitation is such that any reaction back to the abutment has no effect on its ability to shake the vibrating system. Now, we have:

$$m\ddot{u}_r + f\dot{u}_r + ku_r = -m\ddot{U} \tag{3}$$

Motion of support is $U = U_o \cos\omega t$

The *relative* motion between the abutment and the mass is $u_r$ such that:

$$u_r = u - U$$

$$u_r = \frac{U_o r^2 \cos(\omega t - \phi)}{\sqrt{(1-r^2)^2 + (2cr)^2}}$$

$$\tan\phi = \frac{2cr}{1-r^2}$$

$$u = \frac{U_o\sqrt{1+(2cr)^2}\cos(\omega t - \phi + \eta)}{\sqrt{(1-r^2)^2+(2cr)^2}}$$

$$\tan\eta = 2cr$$

The responses *u and* $u_r$ are the Frequency Response Functions. These should be sketched out so that resonance can be defined and the characteristic differences between the different forms of excitation and response [displacement, velocity or acceleration] can be pointed out.

An important, and often overlooked, feature is Transmissibility, *T*, which defines the force transmitted from the vibrating system to the support, and thence to its surroundings. For the case of an excitation of the form *P cos ωt* [see Figure 5] we have that the force transmitted to the support, $F_T$, is given by:

$$F_T = ku + f\dot{u}$$

so $$T = \frac{ku + f\dot{u}}{P}$$

and it can be shown that:

$$T = \frac{\sqrt{1+(2cr)^2}\quad\cos(\omega t - \phi + \theta)}{\sqrt{\left(1-r^2\right)^2+(2cr)^2}}$$

Which is of the same form as the FRF for abutment excitation.

The concept of Transmissibility leads to another demonstration and an introduction to acoustics. I made a box about 500mm cube which was open on one side. It was made from 20mm thick chipboard and lined with 20 mm thick carpet underfelt. The open side was supported on 10mm diameter rubber tube which provided a seal when the box was rested on it. The excitation source was a small electric "buzzer" which sat on a 100 gm mass. Some soft foam and some light coil springs completed the inventory. When held in the hand, the buzzer can usually be heard by the class. However, when placed on a desk, it couples well because the desk acts as a sounding board [as in a violin]. But when we introduce the soft foam or springs between the buzzer and the desk, the sound is much reduced. We now see that soft springs are good for reducing the force transmitted to the desk, which can be seen in the variation of *T* with frequency above the resonance region where $r >> 1$. If the box is placed on the desk with the buzzer inside and isolated from the desk, all is silent. If the foam or springs are removed, the buzzer can now be clearly heard as it is exciting the desk and so by-passing the box. If we now put the buzzer in the box on the foam, all is silent again until the box is tilted with a 50mm or so gap towards the students. Now, the buzzer is clearly heard. So, what has been learned? First and foremost, isolation from the surroundings reduces the energy transmitted. Second, a simple acoustic enclosure is very effective at reducing airborne acoustic propagation, provided the source [buzzer] is isolated so that it cannot by-pass the box. Third, even a small opening in the enclosure easily allows sound to escape.

At the other end of the scale, I showed a rubber block which contained 2mm thick steel sheets which was used as a building support where earthquakes were prevalent. The steel sheets were placed in the horizontal direction; this provided a stiff vertical support but was flexible to the [mainly] horizontal motion which is so dangerous in earthquakes.

### *3.4. Two or more degrees of freedom*

Systems with two or more degrees of freedom [DOF] were also taught. While two degrees of freedom could be solved "by hand", systems with three or more degrees of freedom were tediously complex and best left to a computer. Also, even with 2 DOF, it was difficult to solve the equations if damping was added. A 2 DOF system, without damping, is shown in Figure 7. Note that $m_2$ can either be connected by one spring [$k_2$] or by another which is terminated in an abutment.

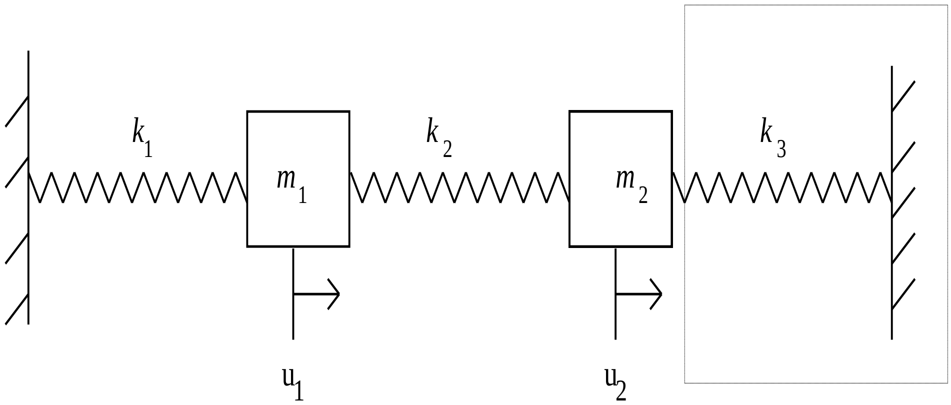



**Figure 7**: Two degrees of freedom

To establish the equation of motion, we consider the forces acting on $m_1$ and $m_2$. From the solution of this equation, we can determine the natural frequencies $\omega_1$ and $\omega_2$, and the corresponding mode shapes. Note that it is still a 2 DOF system if $k_3$ exists or is zero. From the forces acting on $m_1$, we get:

$$m_1 \ddot{u}_1 + (k_1 + k_2) u_1 = k_2 u_2$$

$$\left(m_1 D^2 + k_1 + k_2\right) u_1 = k_2 u_2$$

Rearranging and noting that $D^2 \equiv -\omega^2$, we get the mode shape $u_2/u_1$

$$\frac{u_2}{u_1} = \frac{k_1 + k_2 - m_1 \omega^2}{k_2}$$

Similarly, for $m_2$ we have:

so $$m_2 \ddot{u}_2 = -k_2 (u_2 - u_1) - k_3 u_2$$

or $$m_2 \ddot{u}_2 + (k_2 + k_3) u_2 = k_2 u_1$$

whence, as before,

$$\frac{u_2}{u_1} = \frac{k_2}{k_2 + k_3 - m_2 \omega^2}$$

Equating the mode shapes gives us:

$$m_1 m_2 \omega^4 - \left[ m_1 (k_2 + k_3) + m_2 (k_1 + k_2) \right] \omega^2 + k_2 k_3 + k_3 k_1 + k_1 k_2 = 0$$

Even though this equation is a quartic, it only has 2 real roots, say $\omega_1$ and $\omega_2$. These are the natural frequencies. Dividing the equation in $\omega^4$ by $m_1 m_2$ gives:

$$\omega^4 - \left[\frac{k_2+k_3}{m_2} + \frac{k_1+k_2}{m_1}\right]\omega^2 + \frac{k_2k_3+k_3k_1+k_1k_2}{m_1m_2} = 0$$

If we know $k_1$, $k_2$, $k_3$ and $m_1$, $m_2$, we can determine $\omega_1$ and $\omega_2$. There will be a different mode shape, $u_2/u_1$, at each natural frequency.

Because there are now two natural frequencies, it can be shown that:

$$u_1(t) = A_1\cos\omega_1 t + B_1\sin\omega_1 t + C_1\cos\omega_2 t + D_1\sin\omega_2 t$$
$$u_2(t) = A_2\cos\omega_1 t + B_2\sin\omega_1 t + C_2\cos\omega_2 t + D_2\sin\omega_2 t$$

Where:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \text{1st mode shape}, \quad \frac{C_1}{C_2} = \frac{D_1}{D_2} = \text{2nd mode shape}$$

The constants A, B, C, D are determined from the initial conditions. A typical response of one of the masses to an initial displacement might be of the form:

$$y = 3\cos 3t + 2\cos 7t$$

As shown in Figure 8.

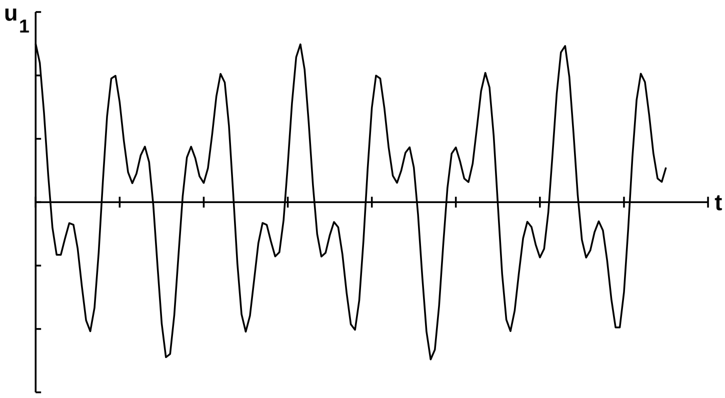



**Figure 8**: Transient response of a two degrees of freedom system

If this signal is electronically filtered, we would see that it has two natural frequencies at 3 and 7 rad/sec as shown in Figure 9.

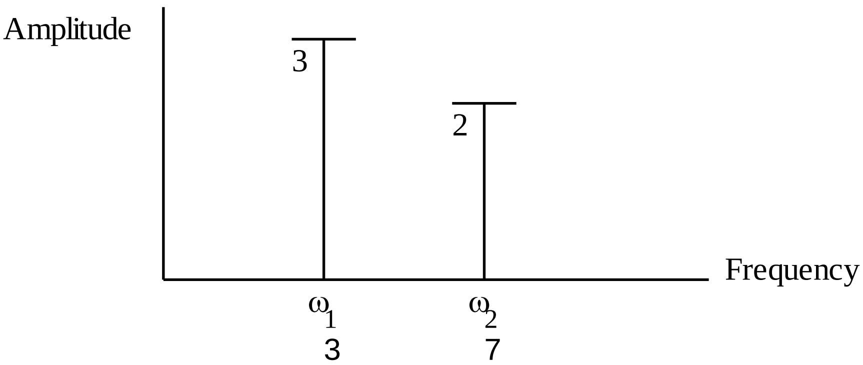



**Figure 9**: Frequency domain representation of a two degrees of freedom system

It is at this point that I introduce another demonstration. I used a block of wood about 25 mm thick into which were glued two steel strips as cantilevers, positioned about 200mm apart. The strips were made of spring steel, 1 mm thick, 10 mm wide, and 250mm long. At the ends was a hard plastic red ball, about 50mm diameter [probably from a snooker table] with a mass of about 150gm. The strips were connected by a pre-tensioned coil spring; mine was about 10mm diameter and used 0.85 mm wire.

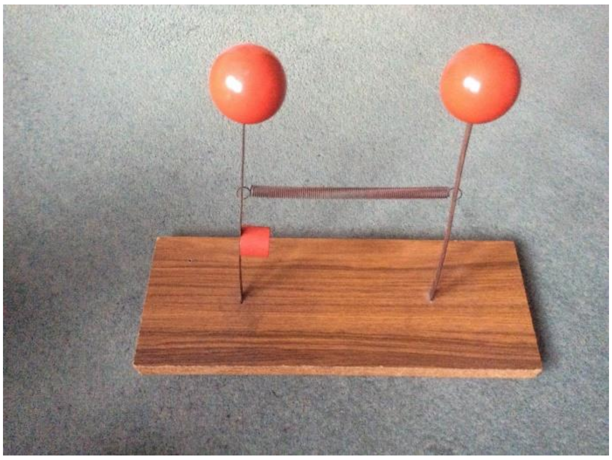


**Figure 10**: Two degrees of freedom demonstration model

Note that my physical model was not as in the diagram above but needed an additional spring of stiffness $k_1$ attached to another abutment on the right. By giving the base a sharp push, the two cantilevers could be made to move together in the first mode of vibration. By holding them apart and releasing, the second mode was demonstrated. By moving the spring to different positions, thereby changing its effective stiffness, it could be shown that the frequency of the first mode was independent of the spring stiffness while that of the second mode was not. But now we come to the interesting part. By attaching a 100gm magnet to one of the strips, displacing and then releasing, it could be shown that energy was transferred from one cantilever to the other and back again such that each in turn might be instantaneously at rest at one time or another or, by suitable positioning of the magnet, only one might come to rest instantaneously. It was easy to move the magnet and to give different initial conditions, creating many ways in which the system could vibrate, all of which could be described by the equations above. A colleague said he could model my system on his computer and demonstrate it on a screen, but it would be much less interesting or convincing to the students.

Forced vibration with a 2 DOF system is straightforward [providing damping is not introduced]. Using $P \cos \omega t$ as the excitation applied to mass 1, it can be shown that the motion of the two masses is given by:

$$u_1 = \frac{P\left(k_2 - m_2\omega^2\right)\cos\omega t}{m_1 m_2\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right)} = \frac{P\left(1 - \omega^2 m_2/k_2\right)\cos\omega t}{k_1\left(1 - \omega^2/\omega_1{}^2\right)\left(1 - \omega^2/\omega_2{}^2\right)}$$

and

$$u_2 = \frac{P\cos\omega t}{k_1\left(1 - \omega^2/\omega_1{}^2\right)\left(1 - \omega^2/\omega_2{}^2\right)}$$

Inspection shows that resonance will occur when the excitation frequency $\omega$ coincides with either of the two natural frequencies. An important result is when $\omega^2 = k_2/m_2$. This defines the detuned frequency where the first mass is at rest while the second mass moves to exactly balance the excitation. In certain cases, this is a very useful phenomenon and can be used in practice with great effect. If damping is present, $u_1$ is never zero but the frequency range over which the vibration amplitude is reduced is widened. The case with damping is sometimes referred to as a vibration absorber. I was consulted by a former student to see if it was possible to control the vibration transmission from a large shaker [using contra-rotating out of balance masses] used to crush refractory powder in a ball mill. The transmitted vibration was causing damage to the walls of the building and suggested solutions were very expensive or unfeasible. Fortunately, the electric motor ran at a constant speed, so by designing my detuner to oscillate at this frequency, the transmitted vibration was reduced almost to zero, much to the amazement of the technicians who ran the ball mill. A year later, it was still working perfectly. Colleagues who worked on machine tool vibration used this same principle [usually with damping] to stop machines "chattering" when the machine tool vibrates so that the cutting tool leaves unwanted marks on the workpiece.

### *3.5. Torsional vibration*

Torsional vibration is equivalent to linear vibration except that

$u \to \theta$

$m \to I$ (moment of inertia)

$k \to$ torsional stiffness

The main complication occurs in geared systems as shown in Figure 11.

The reduction ratio = $n = r_3/r_2 = -\theta_2/\theta_3$

To simplify the solution, we let $I_3$ be zero.

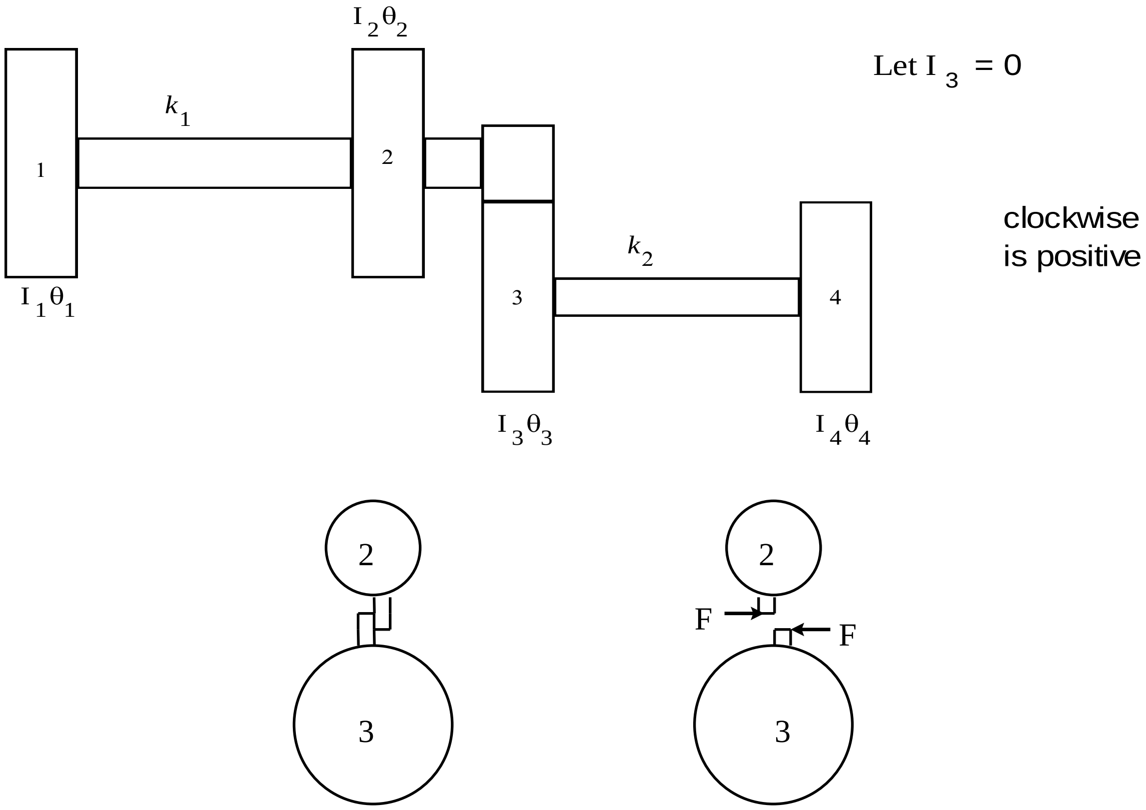



**Figure 11**: Torsional vibrating system showing the interaction of the gear teeth

Considering the moment acting on $I_1$ and applying Newton's second law, we have:

For $I_1$

$$I_1\ddot{\theta}_1 = k_1(\theta_2 - \theta_1)$$

$$\therefore \quad I_1\ddot{\theta}_1 + k_1\theta_1 = k_1\theta_2$$

$$\text{or } (k_1 - I_1\omega^2)\theta_1 = k_1\theta_2$$

Similarly for $I_4$,

$$I_4\ddot{\theta}_4 = k_2(\theta_3 - \theta_4)$$

$$\therefore \quad I_4\ddot{\theta}_4 + k_2\theta_4 = k_2\theta_3$$

$$\text{or} \quad (k_2 - I_4\omega^2)\theta_4 = k_2\theta_3$$

For $I_2$, (anticlockwise)

$$I_2\ddot{\theta}_2 = k_1(\theta_1 - \theta_2) - F.r_2$$

Since $I_3 = 0$, the net torque acting on 3 is zero, so:

$$- F.r_3 + k_2(\theta_4 - \theta_3) = 0$$

We have 5 equations and 5 unknowns, so we can eliminate 4 unknowns to get a single equation in say $\theta_2$, which will give the frequency equation. After some manipulation, we have:

$$I_1I_2I_4\omega^4 - \omega^2\left[k_1I_4(I_1 + I_2) + k_2I_1(I_2 + I_4/n^2)\right]$$
$$+ k_1k_2\left[I_1 + I_2 + I_4/n^2\right] = 0$$

This has two positive roots, $\omega_1$ and $\omega_2$. The mode shapes can be determined by substituting $\omega_1$ and $\omega_2$ in turn into the above equations.

***3.6. Approximate numerical solutions of Rayleigh and Stodola***

Students were introduced to Rayleigh's method for obtaining a numerical estimate of the lowest natural frequency of a structure. Rayleigh's method is based on energy. For a linear system, with no damping, the total energy at any time is constant,

Kinetic Energy + Stored Energy is a constant, i.e. KE + SE = R, so

maximum K.E. = maximum S.E.
(SE = 0) (KE = 0)

For a system of masses $m_1$ $m_2$ ... with displacements $u_1$ $u_2$ ... vibrating at some natural frequency $\omega_i$,

$$\mathrm{KE} = \mathrm{KE}\left[(\text{velocities})^2\right]$$

$$\mathrm{SE} = \mathrm{SE}\left[(\text{displacements})^2\right]$$

or $\quad \mathrm{KE} = \mathrm{KE}\left[\omega_i^2 u^2\right] = \omega_i^2 \mathrm{KE}\left[u^2\right]$

and $\quad \mathrm{SE} = \mathrm{SE}\left[u^2\right]$

so $\quad \omega_i^2 = \dfrac{\mathrm{SE}(u^2)}{\mathrm{KE}(u^2)}$

If $u_1$, $u_2$ etc. are exact, then $\omega_i^2$ is the true natural frequency according to that <u>mode</u> <u>shape</u>. However, if we <u>guess</u> the mode shape, $\omega_i$ will not be exact. Rayleigh's theorem tells us that a reasonable approximation to $u_1$ $u_2$ ... $u_p$ will give a good approximation to $\omega_i$. Three points to note are:

(1) An incorrect guess will be equivalent to applying constraints to the system, leading to an increase in SE. Thus, for the fundamental frequency, Rayleigh will always give a high value of $\omega$. The better the guess, the nearer is $\omega$ to the true natural frequency.

(2) We always use strain energy and not gravitational energy since oscillation about the equilibrium position removes the gravitational term.

(3) A good guess is often the static deflected shape.

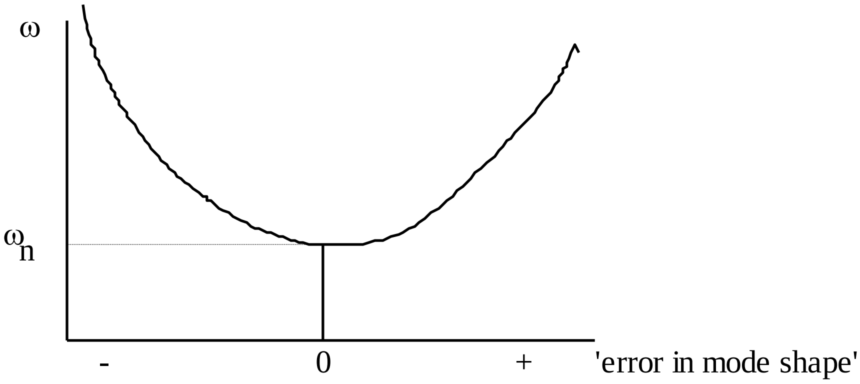



**Figure 12**: Illustration of the frequency error as a function of mode shape with Rayleigh's method

A method of improving the accuracy of the frequency prediction is to use Stodola's iterative method which uses the frequency predicted by Rayleigh's method, together with the equation of motion, to improve the guessed mode shape. This step can be repeated as required. A simple example using the system in Figure 7 with equal masses and springs can produce a very accurate value for the mode shape and frequency in just a few steps.

In year 3, the topics covered were on systems with distributed mass and stiffness, such as bars, beams, and plates. The differential equations were different from those used for the discrete systems taught in year 2, and particular attention was paid to boundary conditions, since these describe how the vibrating system interacted with its surroundings. Because the vibration changes with position as well as time, we need to use partial differential equations. Because of the more complicated mathematics involved, damping was not introduced, nor were forcing functions. The aim was to establish natural frequencies and mode shapes, since these are what are needed in practice. Introducing time functions was by-passed and it was assumed that we always had steady state sinusoidal oscillation. Transient situations will occur, of course, and can be dealt with by using the mathematics developed if so needed.

### *3.7. Axial Vibration*

I started the course with the axial vibration of uniform prismatic bars. In practice, the treatment was restricted to bars which were slender, in which the length/diameter ratio was at least 10. Transverse [radial] motion is caused by Poisson's ratio coupling so the "length" parameter really relates to the wavelength. At higher frequencies, the wavelength becomes shorter so that the simple equations need to be interpreted with care.

A schematic of a uniform bar in axial vibration is given in Figure 13 where the symbols have their usual meaning. We consider an element of the bar at distance *x* from one end; the arrows shown define the positive *x* direction. The stress, strain, and displacement will

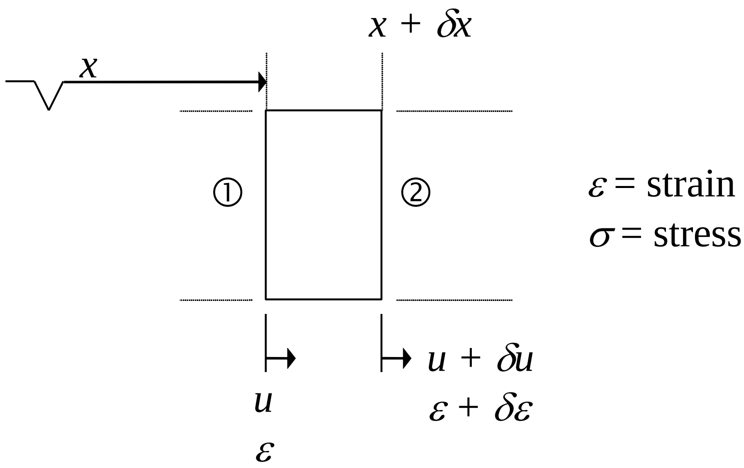



**Figure 13**: Element of a bar in axial vibration

change from position 1 to position 2 as a function of *x*. The normal force *P* acting on face 1 in Figure 13 in the positive *x* direction is given by:

$$-AE\frac{\partial u}{\partial x}$$

Where *E* is Young's Modulus and *A* is the area.

Similarly, the force acting on face 2 in the positive x direction is:

$$+AE\left(\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\delta x\right)$$

By subtracting the forces on face 1 from those on face 2, and noting that a positive value of *∂u/∂x* represents a tensile strain, we can show that the net force acting on the element is

$$=+AE\frac{\partial^2 u}{\partial x^2}\delta x$$

Note that we must now use partial derivatives since *u* is a function of both *x* and *t*.

By applying Newton's second law, we have:

$$AE\frac{\partial^2 u}{\partial x^2}\delta x = A\rho\delta x\frac{\partial^2 u}{\partial t^2}$$

Which, in the limit, gives us the equation of motion for axial vibration,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} \quad (4)$$

It is useful to note that $E/\rho = c^2$ where *c* is the speed of extensional waves in the bar. The general solution of Equation 4 is:

$$u = u(x,t) = u(x).u(t)$$

Where:

$u$ = (P cos$\alpha x$ + Q sin$\alpha x$) (T cos $\omega t$ + U sin $\omega t$)

$\alpha = 2\pi f/c$

$f$ = frequency (Hz)

$\omega = 2\pi f$

The boundary conditions give the constants *P, Q, T, U.* Normally, *T* and *U* are not considered as most cases concern steady state oscillation, but they are there if needed. A bar may be fixed at one end or free. It might also be terminated by a spring or a [point] mass. Apart from the free condition, all the other boundary conditions are impossible to achieve in practice. For instance, a steel bar cannot be attached to a block of infinite stiffness, springs have mass as well as stiffness, and masses cannot be made of a material of infinite density so cannot be "point" masses. However, it is useful to look at the ideal situations as these give a guide to the likely natural frequencies and mode shapes to be expected.

Fixed

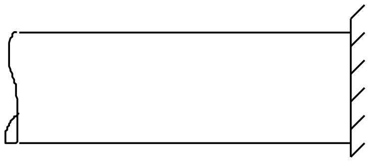



**Figure 14**: Bar terminated in a rigid foundation [difficult in practice]

$$u = 0, \quad \varepsilon = \frac{\partial u}{\partial x} = ?$$

Free


**Figure 15**: Bar free at the end

There is no force, so $\sigma = 0$, so

$$\varepsilon = \frac{\partial u}{\partial x} = 0$$

and $u$ = ?

Mass at the end of a bar

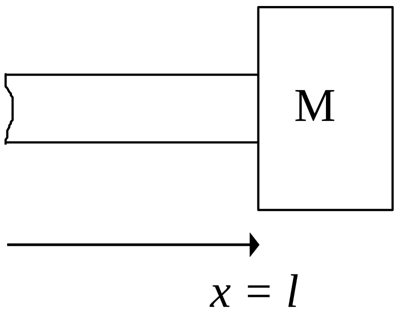



**Figure 16**: Bar with mass at the end

At $x = l$, we need a force to keep M on the bar, so by Newton's 2nd Law

Force = mass x acceleration

!! SIGNS !!

$$\therefore \text{Force} = AE\frac{\partial u}{\partial x}$$

and $$+ve\frac{\partial u}{\partial x} \equiv \text{tension}$$

The force on the mass is in the negative direction of *x* which gives:

$$AE\left(\frac{\partial u}{\partial x}\right)_l = -M\left(\frac{\partial^2 u}{\partial t^2}\right)_l = +M\omega^2 u_l$$

If we have a bar fixed at one end with a mass at the other, we have:

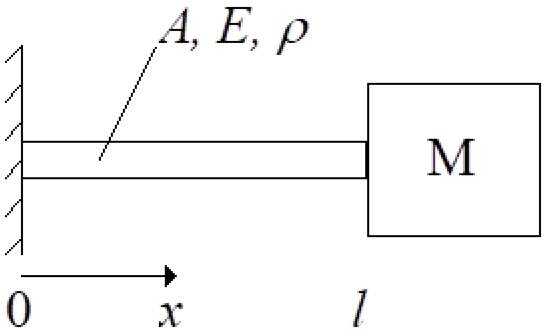



**Figure 17**: Bar fixed at one end and free at the other

$x = 0, \qquad u = 0$

and at $x = l$

$$AE\left(\frac{\partial u}{\partial x}\right)_l = +M\omega^2 u_l$$

which leads to:

$$\tan \alpha l = \frac{AE\alpha}{M\omega^2}$$

But $A\rho l = m$ = mass of the bar and it can be shown that the frequency equation is:

$$\alpha l \tan \alpha l = \frac{m}{M}$$

Where $\alpha = \omega / c$

This equation cannot be solved explicitly. It is called a transcendental equation and has to be solved graphically as shown in Figure 18. Note that as we cannot have a point mass, the solution is only approximate. If $M = 0$, $m/M = \infty$ and the solution gives $\pi/2$, $3\pi/2$, etc which is for a fixed/free bar. If $m/M = 1$, the solution is just greater than $\pi/4$ for the first mode.

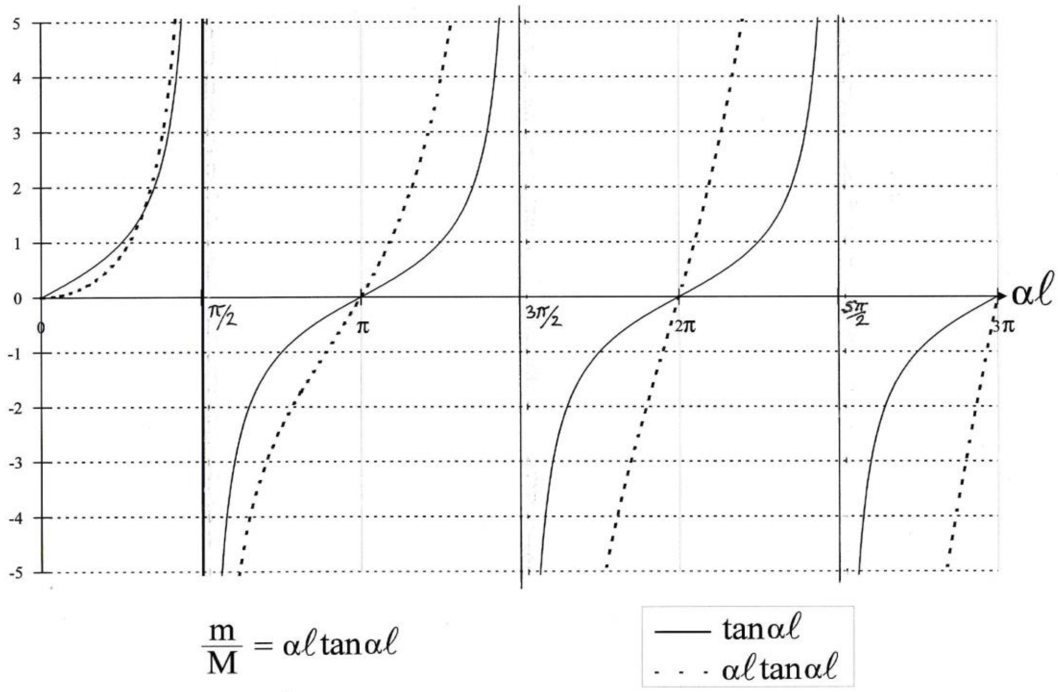



**Figure 18**: Graphical solution to the transcendental equation

Spring

If the bar is terminated by a spring, we have:

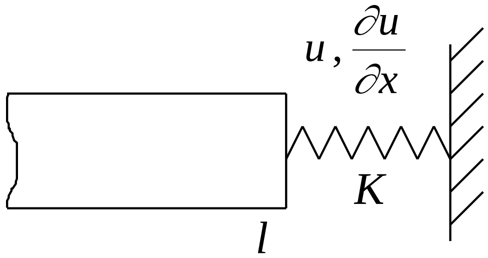



**Figure 19**: Bar with a spring termination

The force in the bar at *x* = *l* and the displacement of the bar at *x* = *l* is the same as in the spring, so [being careful of the signs]:

$$\therefore AE\left(\frac{\partial u}{\partial x}\right)_l = -K(u)_l$$

Two bars connected together

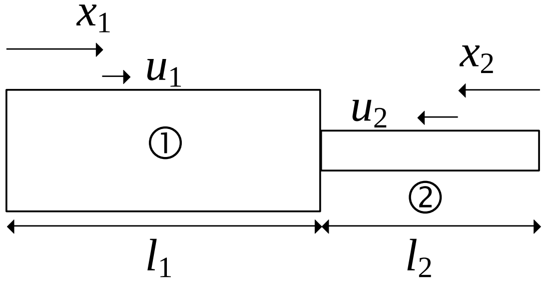



**Figure 20**: Two bars connected together

There must be a continuity of force at the junction, so:

$$A_1E_1\varepsilon_1 = A_2E_2\varepsilon_2$$

We also have compatibility of displacements, so, at the junction,

$$(u_1)_{l_1} = -(u_2)_{l_2}$$

Note that it is MUCH better to specify $x_1$ and $x_2$, $u_1$ and $u_2$, in opposite directions as in Figure 20. If you don't believe me, try the alternative!

Because $u_1$, and $x_1$ are positive in the same direction, as are $u_2$ and $x_2$, then:

$$\frac{\partial u_1}{\partial x_1} + ve, \frac{\partial u_2}{\partial x_2} + ve$$

And both indicate tension. At the join, we have:

$$A_1E_1\left(\frac{\partial u_1}{\partial x_1}\right)_{l_1} = A_2E_2\left(\frac{\partial u_2}{\partial x_2}\right)_{l_2}$$

Such a system, in which the two half bars are of equal length, is used in a resonant wave guide for high power ultrasonics. It can be shown that the ratio of the amplitudes of vibration at the two ends are inversely proportional to the areas of the bars. At resonance, each of the two bars is a quarter wavelength long. In reality, the conditions at the join with respect to force are mathematically incorrect and the sharp change in section is smoothed by a radius to avoid fatigue failure. Nonetheless, it works, and I used such a device resonating at 11.6 kHz for over two years when measuring the damping of metals at cyclic stresses up to their fatigue limit.

**Figure 21**: Waveguide for high power ultrasonics

Torsion of bars

Torsional vibration provides a lot of conceptual problems for students. The basic equations are the same as in axial vibration, but the sign convention is less obvious and requires very careful attention.

| | | |
|---|---|---|
| J | = | polar 2nd moment of area |
| | = | $\pi D^4/32$ |
| G | = | shear modulus = $E/2(1+v)$ |
| $v$ | = | Poisson's ratio |
| $\rho$ | = | density |

$$\text{Torque} = T = GJ\frac{\partial\theta}{\partial x}$$

$\theta$ = angular displacement.

The outward going normal in + *ve x* direction = positive face.
Clockwise rotation and clockwise torque viewed in + *ve x* direction are positive

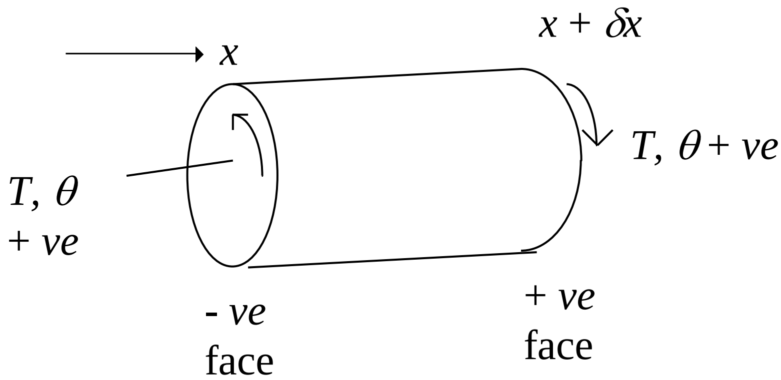



**Figure 22**: Element of bar in torsional oscillation

At *x*, the torque acting on the element is $T = -GJ\dfrac{\partial\theta}{\partial x}$

At $x + \delta x$, the torque is:

$$T + \delta T = +GJ\frac{\partial\theta}{\partial x} + \frac{\partial T}{\partial x}\delta x$$

So the net torque acting on the element is:

$$(T + \delta T) - T = GJ\frac{\partial^2\theta}{\partial x^2}\delta x$$

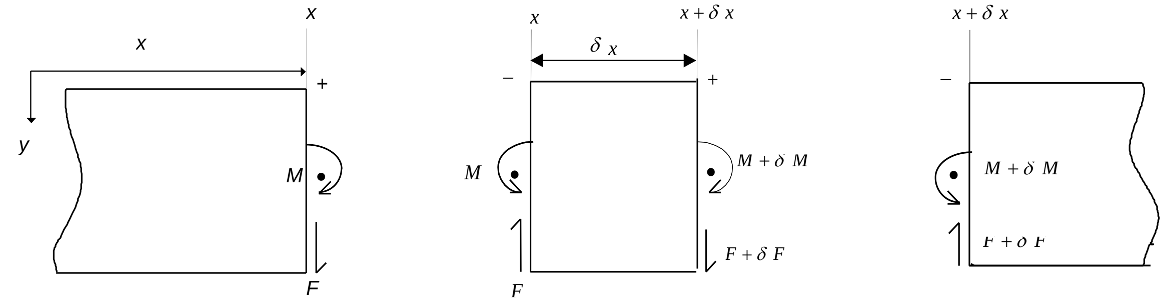
Using Newton's second law in torsional rotation and noting that the moment of inertia of the element is *Jρ.δx* gives our equation of motion:

$$\therefore \left(\frac{G}{\rho}\right)\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2}$$

which is very similar to that for axial vibration.

<u>Flexural vibration of beams</u>

Flexural vibration is somewhat more complicated than axial vibration. I used the Bernoulli-Euler theory for the bending of beams in which plane sections remain plane and perpendicular to the neutral axis. It is necessary to adopt and rigidly adhere to a strict and consistent sign convention. For this reason, the sign convention was attached to the equation sheet issued to the students early in the course and made available to them in the exam. Other conventions exist and it is a matter of choice which is used. It is essential that positive and negative faces are clearly defined; in our convention, a positive face is where the outward going normal is in the positive *x* direction. The displacements *u, v,* w, are in the *x, y, z* directions. We have defined the *y* direction as positive downwards and the *z* direction as directed into the page.




$$M = EI\frac{\partial^2 v}{\partial x^2}$$

$$F = -EI\frac{\partial^3 v}{\partial x^3}$$

**Figure 23**: Sign convention and definition of the shear force, *F*, and the bending moment, *M*, for flexural vibration

From Figure 23, the net force on the element is:

$$\delta F = [\partial F/\partial x]\,\delta x$$

The mass of the element is $\rho A\,\delta x$ so by using Newton's second law and simplifying we have the equation of motion for the vibration of a uniform beam:

$$\frac{\partial^4 v}{\partial x^4} = \alpha^4 \frac{\partial^2 v}{\partial t^2}$$

where $\rho A\,\omega^2 = EI\alpha^4$

and *I* is the second moment of area.

Appropriate mathematical manipulation leads us to the solution of this partial differential equation of the form:

$$v = (P\cos\alpha x + Q\sin\alpha x + R\cosh\alpha x + S\sinh\alpha x)(T\cos\omega t + U\sin\omega t)$$

Where the constants PQRSTU are defined by the boundary conditions and

$$\omega = 2\,\pi F$$

and *F* is the frequency of oscillation. [I have *F* for frequency and *F* for shear force as there is an unfortunate clash of accepted nomenclature]. *T* and *U* will only be called on if there is a need for a transient vibration solution, but this will only be a tiny minority of cases as we are mainly interested in mode shapes and natural frequencies. Because we have 4 constants PQRS, we need 4 equations which can usually be obtained from the boundary conditions.

At a free end, the bending moment, *M*, and the shear force, *F*, will be zero, so $\partial^2 v/\partial x^2 = 0$ and $\partial^3 v/\partial x^3 = 0$

At a fixed end, there is no deflection and no slope, so $u = 0$ and $\partial v/\partial x = 0$

At a pinned [simply supported] end, there is no deflection and no moment, so $u = 0$ and $\partial^2 v/\partial x^2 = 0$

Of these boundary conditions, the free end is easily obtained, but the other two are difficult, if not impossible, to achieve in practice. For a beam vibrating freely (no end restraints), the frequency equation is:

$$\cos \alpha l \cosh \alpha l = 1$$

The solution of this equation cannot be obtained explicitly and needs to be solved numerically. Figure 24 gives the mode shapes and the nodal positions for the first five modes of vibration. The constant B is used to define the natural frequency using the equation:

$$F\left(Hz\right) = \frac{Bdc}{\ell^2}$$

where $\ell$ is the length of the beam, *d* is its thickness, and $c = \sqrt{E/\rho}$.

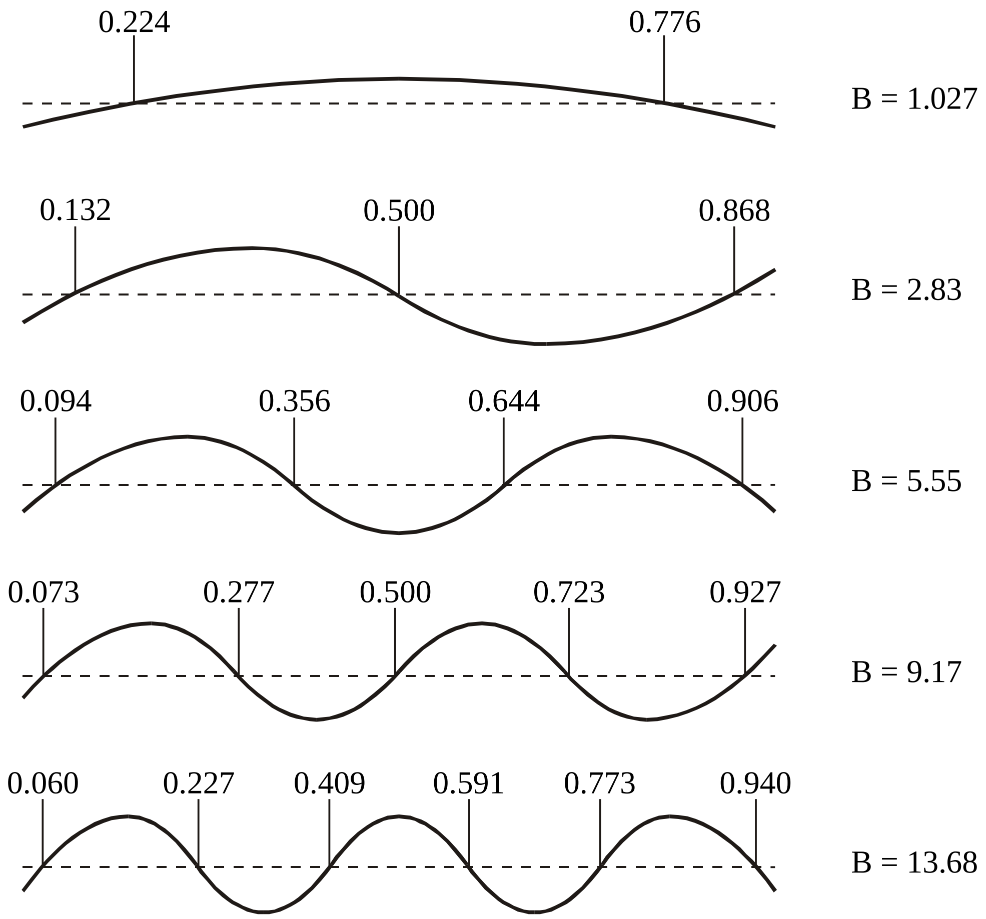



**Figure 24**: Mode shapes, nodal positions, and frequency numeric for the first 5 modes of flexural vibration of a uniform free-free beam

Other boundary conditions, such as a mass at the end of a beam or some position along it, can be incorporated by using the bending moment and shear force terms from the bending of the beam together with Newton's second law. But note that since point masses do not exist, the results will only be approximate. It is very important that signs are carefully observed, and it is understood what [shear force or bending moment] is acting on which face. I give below an example for a mass at the end of a beam. In Figure 23, consider the element of length $\delta x$ as a point mass. The positive shear force on the negative face [of the mass] is acting in the negative *x* direction. Using Newton's second law, we can write:

$$-F = m\, \partial^2 v/\partial t^2$$

But since $F = -EI\,[\partial^3 v/\partial x^3]_l$ and $\partial^2 v/\partial t^2 = -\omega^2 v_l$

Then, $EI\,[\partial^3 v/\partial x^3]_l = -m\,\omega^2 v_l$

I also introduced the class to the concepts of additional terms to the Bernoulli-Euler equation for a flexurally vibrating beam to allow for shear and rotary inertia. These terms are needed if the beams are thick or for predicting the frequencies of higher modes.

Plate vibration

A vibrating plate is a two-dimensional system. Algebraic solutions of the equation of motion can only be determined for simple shapes, such as rectangles and circles, and with certain types of boundary conditions. For a rectangular plate in the *x-y* plane, the equation of motion is:

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{12\rho\left(1-v^2\right)}{Eh^2}\frac{\partial^2 w}{\partial t^2}=0$$

where *w* is the deflection in the *z* direction. This differential equation can only be solved explicitly for certain conditions. If the plate sides are simply supported (hinged), then the solution is:

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(T\cos\omega t+U\sin\omega t\right)$$

where *a* and *b* are the side lengths in the *x* and *y* directions, and *m* and *n* define the number of half waves in the *x* and *y* directions. It is very difficult to create simply supported or clamped edges in practice, although free edges are easily achieved.

At this point, I showed a Rolls-Royce fan blade to show the challenge of predicting the natural frequencies of a real structure and thanked the inventors of fast digital computers for enabling the solution via Finite Element Analysis. I also demonstrated a musical saw [a common wood saw], which has some interesting characteristics. As a tapered flat sheet of steel terminated in a wooden handle, it has a variety of natural frequencies. The saw has 3 edges free and the fourth has an uncertain termination and is heavily damped because of the handle and how it is held. However, when the blade is bent into an “S” shape, the damping is very small and it is possible to play musical notes when stroking one of the long free edges with a ‘cello bow, or even a wooden rod. On one occasion, the students filmed me playing the saw [sitting on a chair which was on a table] and sent it to their non-engineering friends to show that engineering lectures were not boring!

Rayleigh’s method for continuous systems

Continuous systems, such as rods, beams, and plates, present some challenging, and often impossible, problems to solve by conventional algebra. Numerical solutions based on Rayleigh’s method are particularly useful in predicting natural frequencies, especially the fundamental frequency. The same principles apply as with lumped systems. In particular, the better the guessed mode shape, the more accurate is the frequency prediction. Knowledge of Rayleigh’s method can be tested in examinations, so it is worth teaching both for knowledge and assessment purposes.

Using a fixed-free bar in axial vibration as an example, it can show that by using a ¼ sine wave as the mode shape, the exact natural frequency is obtained. This is not surprising but is a useful demonstration of how Rayleigh’s method works. But a linear mode shape where the displacement is proportional to *x* gives an error of only 10.25%, even though it violates the boundary condition [zero strain] at the free end of the bar. On the other hand, using the statically deflected shape, in which the bar hangs under gravity from the fixed end, gives an error of only 0.635% as this mode shape satisfies the boundary conditions at both ends. Of course, the algebraic computation is much greater than for the linear option. But whereas the case with a mass on the end of a bar needed and graphical/numerical solution, an example with a linear mode shape and a mass equal to that of a bar could be solved in a few lines with an error of less than 1%.

For flexural vibration, a cantilever was used as an example. Note that there are now 4 boundary conditions to satisfy. A variety of mode shapes was used. The statically deflected shape gave an error of less than 1%, a cosine-based shape had an error of 1.5% while a parabola gave an error of over 25%.

With plates, boundary conditions other than all simply supported generally have to be solved numerically using beam functions and Rayleigh's method. These days, computer solutions using finite element analysis are normally used to obtain natural frequencies and mode shapes. In practice, circular plates supported on razor blades are the nearest real situation to the mathematics. Using such a support system for rectangular plates poses problems of rotation at the corners which can be partially solved by omitting the supports near to the corners.

Non-linear vibration

A topic which is important when it comes to testing resonant systems concerns non-linearity. This has been extensively covered by many authors and quickly slips into complex mathematics if other than a qualitative system is considered. In effect, there are softening springs and stiffening springs. With softening springs, the resonant frequency decreases with amplitude, while the opposite is true for stiffening springs. With a linear system, there is only one response amplitude for a given frequency. However, in non-linear systems, depending on the level of damping there can be three possible amplitudes near to resonance. As the driving frequency is increased towards resonance for a softening spring, the amplitude of vibration will suddenly increase [jump] from A to B as shown in Figure 25, and then decrease without reaching the resonant amplitude. When the frequency is decreased from above resonance, the resonant amplitude will be reached and then decrease as the frequency is reduced before there is a sudden jump downwards from C to D. The part of the response curve from A to C can never be found in practice. A stiffening spring slopes the other way and also gives the jump phenomenon. For what it is worth, one of the few laboratory practicals I can remember from undergraduate days concerned the flexural vibration of a simply-supported beam which was excited at its mid point by an out of balance mass driven by an electric motor. Because the "simple supports" were very firmly clamped knife edges, deflection of the beam induced tensile forces which gave a stiffening characteristic. I spent much of my Easter vacation trying to find out why my experimental results did not conform to the expected Frequency Response Function for a linear system and to explain what had happened. Frustrating, but I learned a lot!

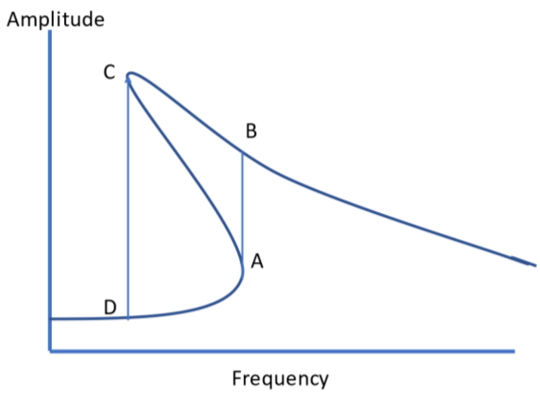



**Figure 25**: Frequency response function for a softening spring

Beyond a simple, qualitative description of nonlinearity and how it affects the resonant response, the algebra ascends [descends?] into hideous complexity which is completely unjustified for an undergraduate course.

Examples classes

Most courses will provide questions following the lectures so that students can test themselves on their understanding our the course.

In year 2, there was a one-hour examples class to support each lecture. In these classes, which usually followed the lecture, the students were issued with sheets of questions which were typical [usually actual] exam questions. However, it is important to grade the questions so that the first few are rather easier to give students confidence in tackling the paper. Attendance at the classes was optional and no records were kept on attendance. I was always at these classes to provide help to the students, two or three minutes being enough for me to see a difficulty and to help resolve it. From time to time, I tried using postgraduates [PhD students or post docs] to assist me, but I found that they spent too much time and often caused more confusion. As the classes were rarely more than 30 students [optional attendance], I preferred to be the lone assistant. There was, of course, a significant amount of self-help between the students themselves. In year 2, the numerical answers were given out, but not the worked solutions. A set of typical examples sheet is given in the Supplementary Information: [SI_Continuous_systems_example_sheets,
SI_Lumped_parameter_Introductory_Example_sheets_1A&2A,
SI_Lumped_parameter_Examples_sheets].

In year 3, the Department did not timetable examples classes, but I was available in my office to help at a specified time. I gave out worked solutions but only much after the question sheets. I found that if such solutions were made available with the question sheets, there was a tendency to skim through the solution but not to understand the basics. As always, the students have a wide range of ability and motivation, and have their own agendas and timescales. That's life, but sometimes people have to be protected from themselves.

For both courses, the exams consisted of a three hour paper following completion of the lecture course. The students were not allowed books or notes but were issued with the same crib sheet they had been working with during the year. Exam papers from previous years were available and students were issued with the answers and some worked solutions.

Laboratory Classes

One of my professors, G F C Rogers, a thermodynamicist who wrote a well-known book with Y R Mayhew, challenged me when I joined the Department to introduce at least one new laboratory experiment each year. Rogers was a great believer in the benefit of laboratory experiments in developing the understanding of engineering principles. There were many discussions as to whether the laboratory classes should come after the relevant lecture or before it. The conclusion was that there were benefits both ways and realistic problems of space, equipment, and the timetable meant that some students had the lectures before, and some after and it did not seem to make a lot of difference in the long run.

To support my vibration lectures, I created a series of laboratory classes [most of the other courses did the same]. These were to help the students to see the lectures from a new perspective. The classes were staffed mainly by PhD students [from my own group] but I had a wandering wizard role to check that all was going smoothly and to add a few words of [I hope] enrichment. Lab sheets were issued to help the students and to guide them into what they might discover from their experience. The students worked in groups of 4 as there was only limited equipment available. While the lazy student might sometimes slip into the background, it was excellent practice for team working. A couple of typical lab sheet is given in the Supplementary information for each part of the course [SI_2nd_year_vibrations_lab_2DoF_sheets and SI_3rd-year-vibrations-lab-plates_sheets].

At one stage, I added an "applied" part to the standard experiment. In one case, I had a Mini exhaust system, excited by a rotating out of balance device, which the students examined for resonances and where it was best to mount the rubber suspension so as to minimise transmission to the vehicle body; it turned out that the manufacturer had got it right. In another example, two car doors were examined for vibration response. One door was just the metal [body in white] and the other was fully trimmed inside. Timetable constraints eventually saw the end of these additions…

Sadly, academic time is increasingly being consumed by the demands of research and pointless administration. Also, undergraduate laboratories need space which is not used for the whole year. Consequently, undergraduate laboratory classes are being slowly dropped from the timetable in many universities. We always need to change with the times and my own experimental programme was new once and often replaced earlier experiments. The laboratory classes are an essential part of bringing an understanding of the mathematical content of the lecture courses. The course lecturer cannot run each experiment, but must be seen, even as a wandering wizard, several times during the class.

## 4. Conclusions

I hope the reader will find the above useful as a basis for his/her own teaching of a course in vibration. My lecture course was developed over many years and you see above the finished product. It fitted into the time allowed by the timetable and was aimed at stretching the intellect of the students, while not trying to blind them with difficult mathematics. You can add to it or subtract from it to suit your enthusiasm, experience, and the time available. But, above all, you must work to enjoy your presentation and always give it 100% effort. Do not be afraid to deviate and recount [briefly] some relevant experience or to tell a joke. Yes, I did sit in a bar in New Orleans watching an out of balance ceiling fan and wondering how I could make it into an exam question, and the Ogden Nash contribution to the compatibility of forces and displacements is left to your literary research.

**Supplementary information: SI_Crib sheets_2_and_3**

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Department of Mechanical Engineering

Notes which may be used in the Vibrations 2M examination.

$$\omega\,(\mathrm{rad/sec}) \quad = \quad 2\pi F\,(\mathrm{Hz})\,(\mathrm{cycles/sec}).$$

**Free vibration:**

$$m\ddot{u} + f\dot{u} + ku = 0$$

$c < 1$ , $$u = e^{-\omega_n ct}\left(A\cos\omega_n t\sqrt{1-c^2} \quad + \quad B\sin\omega_n t\sqrt{1-c^2}\right)$$

$\omega_n\sqrt{1-c^2}$ = damped natural frequency

$c = 1$ , $$u = (A+Bt)e^{-\omega_n t}$$

$c > 1$ , $$u = e^{-\omega_n ct}\left(Ae^{\omega_n t\sqrt{c^2-1}} \quad + \quad Be^{-\omega_n t\sqrt{c^2-1}}\right)$$

$\omega_n = \sqrt{k/m}$ = undamped natural frequency

$c = f/f_c$ = proportion of critical damping

$$f_c = 2\sqrt{km}$$

$\delta = \dfrac{1}{n}\log_e\left(\dfrac{u_1}{u_{n+1}}\right) = \dfrac{2\pi c}{\sqrt{1-c^2}}$ = logarithmic decrement.

**Forced vibration:**

Force on mass

(i) $$m\ddot{u} + f\dot{u} + ku = P\cos\omega t$$

$$u = \frac{P\cos(\omega t - \phi)}{k\sqrt{(1-r^2)^2 + (2cr)^2}}$$

$$\tan\phi = \frac{2cr}{1-r^2} \quad , \quad r = \frac{\omega}{\omega_n}$$

Abutment excitation

(ii) $m\ddot{u}_r + f\dot{u}_r + ku_r = -m\ddot{U}$

Motion of support is $U = U_o \cos \omega t$

$$u_r = u - U$$

$$u_r = \frac{U_o r^2 \cos(\omega t - \phi)}{\sqrt{(1-r^2)^2 + (2cr)^2}}$$

$$\tan\phi = \frac{2cr}{1-r^2}$$

$$u = \frac{U_o \sqrt{1+(2cr)^2} \cos(\omega t - \phi + \eta)}{\sqrt{(1-r^2)^2 + (2cr)^2}}$$

$\tan \eta = 2cr$

**Two-degrees-of-freedom:**

Transient:

$$u_1(t) = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t + C_1 \cos\omega_2 t + D_1 \sin\omega_2 t$$
$$u_2(t) = A_2 \cos\omega_1 t + B_2 \sin\omega_1 t + C_2 \cos\omega_2 t + D_2 \sin\omega_2 t$$

where $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \text{1st mode shape}, \ \frac{C_1}{C_2} = \frac{D_1}{D_2} = \text{2nd mode shape}$

Forced, Steady State:

$$U_1 = \frac{A_1}{1-\left(\frac{\omega}{\omega_1}\right)^2} + \frac{C_1}{1-\left(\frac{\omega}{\omega_2}\right)^2}$$

$$U_2 = \frac{A_2}{1-\left(\frac{\omega}{\omega_1}\right)^2} + \frac{C_2}{1-\left(\frac{\omega}{\omega_2}\right)^2}$$

where $A_1 / A_2 = \text{1st mode shape}, \ C_1 / C_2 = \text{2nd mode shape}$

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**Notes which may be used in the Vibrations 3M examination.**

Axial Vibration

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$c^2 = \frac{E}{\rho}$$

$u$ = (P cos$\alpha x$ + Q sin$\alpha x$) (R cos $\omega t$ + S sin $\omega t$)

$\alpha = 2\pi f/c$

$f$ = frequency (Hz)

$\omega = 2\pi f$

Torsional Vibration

$$\text{Torque} = GJ\frac{\mathrm{d}\theta}{\mathrm{d}x}$$

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r}$$

$E = 2G(1 + v)$

Flexural Vibration

$$\frac{\partial^4 v}{\partial x^4} = -\frac{\rho A}{EI}\frac{\partial^2 v}{\partial t^2}$$

$$\frac{\rho A\omega^2}{EI} = \alpha^4$$

$v$ = (P cos$\alpha x$ + Q sin$\alpha x$ + R cosh$\alpha x$ + S sinh$\alpha x$) (T cos$\omega t$ + U sin $\omega t$)

Strain Energy

| | | |
|---|---|---|
| Direct, | $W = \sigma^2/2E$ | per unit volume |
| Shear, | $W = \tau^2/2G$ | per unit volume |
| Bending, | $W = M^2/2EI$ | per unit length |

Plate Vibration

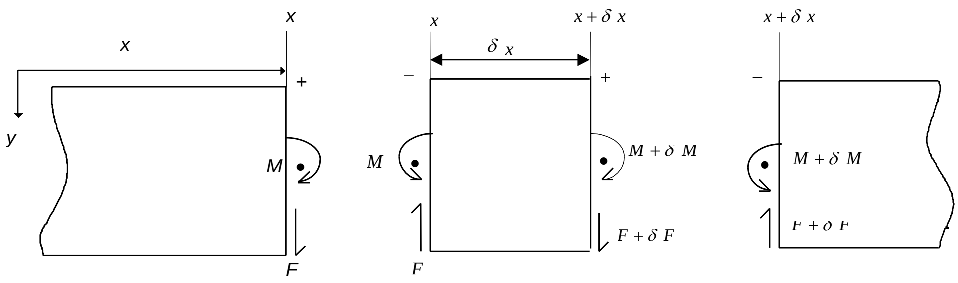
$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{12\rho(1-v^2)}{Eh^2}\frac{\partial^2 w}{\partial t^2} = 0$$

For a simply supported plate of sides a and b,

$$w = W \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\omega t$$

Sign Convention for Flexure

Positive forces and moments acting on positive and negative faces




$$M = EI\frac{\partial^2 v}{\partial x^2}$$

$$F = -EI\frac{\partial^3 v}{\partial x^3}$$

**Supplementary information: SI_Continuous_systems_example_sheets**

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APPLIED MECHANICS - Axial Vibration

Tutorial Sheet 1

1. An axially-vibrating system consists of $n$ rigid masses, $m$, connected by $n$ - 1 springs each of stiffness $k$. If $n >> 1$, this system may be modelled as a uniform bar of length $l$ and cross-sectional area $A$ made from a material of density $\rho$ and Young's modulus $E$.

   An ore-train consisting of 100 trucks, each weighing 30 tonnes is close-coupled such that the buffers are in a state of compression. Each truck has a pair of buffers at each end. Each buffer has an effective spring stiffness of $1.2 \times 10^6$ N/m.

   Assuming that the buffers have sufficient pre-compression that they do not separate, calculate the fundamental longitudinal natural frequency of the system.

   (**Ans**: $3.1623 \times 10^{-2}$ Hz)

2. (Fig. Q2) A steel bar of length 250 mm and radius 10 mm is vibrating in its fundamental axial free/free mode of vibration, such that the amplitude at a free end is 0.1 mm. Calculate the frequency of vibration and the cyclic axial strain amplitude at the mid-point of the bar.

   The bar is horizontal and it is observed that if a loop of light wire is put on the bar near either end, as shown in Fig. Q2, it moves along the bar and stops at the mid-point. Explain why this should happen.

   For steel, E = 210 GPa, $\rho = 7.8 \times 10^3$ kg m$^{-3}$
   Poisson's ratio = 0.29

   The acceleration due to gravity is 9.81 m s$^{-2}$.

   (**Ans**: 10.38 kHz, $1.257 \times 10^{-3}$)

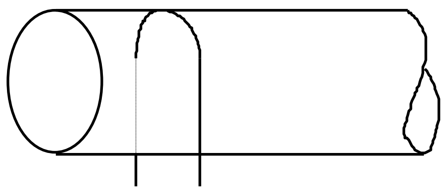
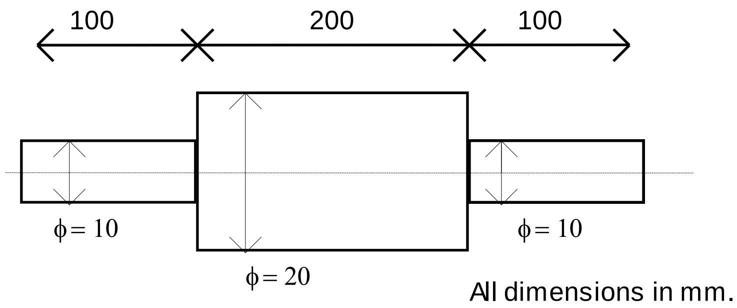


Fig. Q2

3. Calculate the fundamental axial frequency of the free-free steel bar sketched below. The bar is stepped, as shown, and has a circular cross-section.




For steel, E = 210 GPa, $\rho$ = 7.8 x $10^3$ kg $m^{-3}$
You may neglect Poisson's ratio effects.

(**Ans**: 9143 Hz)

4. The stepped, steel bar shown in Fig. Q4 vibrates in its fundamental axial mode. Determine the frequency. The velocity of sound in steel may be taken as 5189 m/s and lateral motion may be neglected. Sketch the mode shape.

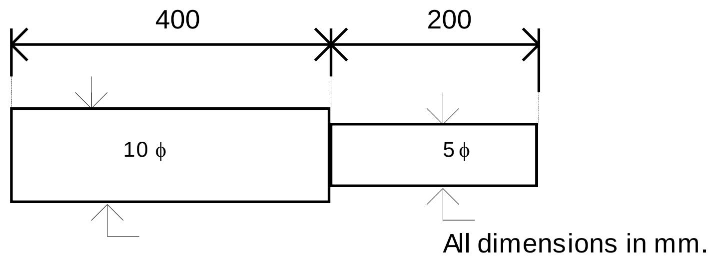



Fig. Q4

(**Ans**: 5157 Hz)

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APPLIED MECHANICS - Flexural Vibration

Tutorial Sheet 2

1. Determine the frequency equation of a uniform beam which is simply-supported at the ends and vibrates in flexure.

   A footbridge has a simply-supported span of 10 m and is carried on two rolled steel joists, each of which weighs 67 kg $m^{-1}$ and has a second moment of area of 2.64 x $10^{-4}$ $m^4$. The platform of the bridge weighs 200 kg $m^{-1}$ and makes no contribution to the stiffness. Determine the lowest frequency of bending vibration (in the vertical plane) of the bridge.

   $E_{steel}$ = 207 GN $m^{-2}$.

   (**Ans:** 8.9856 Hz)

2. Derive the frequency equation for the transverse vibration of a uniform beam of length *l* with one end built in and the other simply supported as shown in Fig. Q2.

   If the transverse stiffness of the simply supported end were finite rather than infinite, would you expect the fundamental frequency to be increased or reduced (argue physically, not by repeating the problem with different boundary conditions)?

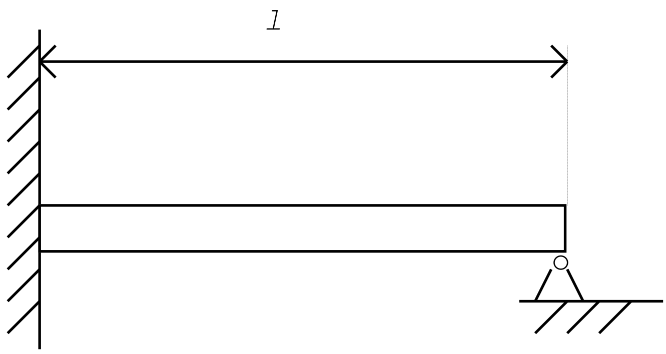


Fig.Q2

   (**Ans:** tan $\alpha l$ = tanh $\alpha l$)

3. The transverse vibration model of a portal frame milling machine is shown in Fig. Q3. The columns are pin-jointed at one end and bolted rigidly to a stiff, heavy cross beam at the other end. The cross beam has twice the mass of each column and prevents any significant rotation of the column end faces.

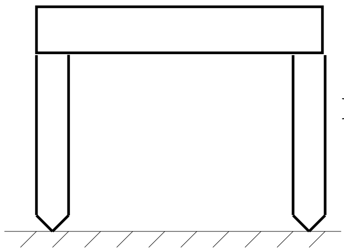


Fig. Q3

Derive the frequency equation for the structure and determine the fundamental natural frequency. Each column has the parameters ascribed to it in Fig. Q3.

(**Ans:** $2 = \alpha l(\tan \alpha l - \tanh \alpha l)$; $\alpha l = 1.19$; $\omega = \frac{1.416}{l^2}\sqrt{\frac{EI}{\rho A}}$ )

4. A helicopter rotor blade may be regarded as a uniform beam which is simply supported at one end and free at the other. The equivalent beam is 4.57 m long with a second moment of area of $8.33 \times 10^{-8}$ $m^4$ and a cross-sectional area of $2.9 \times 10^{-3}$ $m^2$. The rotor blade is made of an aluminium alloy for which Young's modulus is 71 GPa and the density if $2.63 \times 10^3$ $kg/m^3$.

   Find the natural frequencies of transverse vibration in the first and second modes.

   (**Ans:** 3.272 Hz, 10.603 Hz.)

5. A cantilever is driven in flexural vibration by an electrical coil mounted at its free end and moving in the field of a magnet. The coil has been so designed that its mass is sufficiently small that it may be neglected but unfortunately this resulted in the moment of inertia of the coil being by no means negligible.

   For the beam, Young's modulus is *E*, the density is $\rho$ its length is *l* and the second moment of inertia is *I*. The moment of inertia of the coil about an axis through its centre of gravity (which coincides with the end of the beam) perpendicular to the plane of vibration is *J*.

   Determine the frequency equation for this system. If the frequency equation of a cantilever with a completely free end is

   $$1 + \cos \alpha l \cosh \alpha l = 0$$

   and this has as its first solution that $\alpha l = 1.875$, indicate whether the addition of the coil will lead to an increase or a decrease in fundamental frequency of the cantilever.

   (**Ans**: $\frac{EI\alpha}{J\omega^2}\left(1 + \cos \alpha l \cosh \alpha l\right) = \sin \alpha l \cosh \alpha l + \cos \alpha l \sinh \alpha l$ )

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APPLIED MECHANICS - Rayleigh

Tutorial No. 3

1. A two-stage rocket has been modified to place a large space-laboratory in orbit. The payload may be regarded as a rigid mass attached to the second stage, as shown in Fig. Q1. The mass m of the payload is equal to that of the second stage.

Use Rayleigh's method to determine the fundamental frequency of axial vibration after lift-off. Credit will be given for using a realistic mode shape.

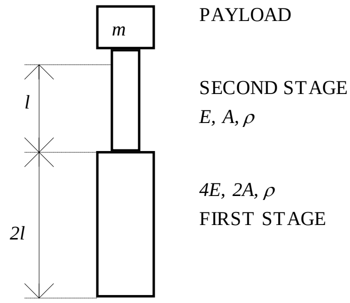


FIG. Q1

(**Ans:** Exact solution is $\omega_1 = 0.97\, c / \ell$ rad / s
where $c = \sqrt{E/\rho}$ )

2. A ship at sea can vibrate in several modes. One mode is usually laterally as a free-free uniform beam. Using Rayleigh's method, determine the fundamental frequency of lateral vibration for a free-free uniform beam of length $\ell$ which has a flexural rigidity *EI* and a uniform weight of $\mu$ per unit length. The lateral deformation can be described approximately by the equation

$$v = b\left(3 \sin\left(\frac{\pi x}{\ell}\right) - 2\right)$$

where *b* is the maximum lateral deflection at mid ship.
Sketch the mode shape.

(**Ans**: Exact solution is: $\omega = 22.37 \sqrt{\frac{EI}{\mu \ell^4}}$ rad / s

Rayleigh → $\omega = 22.57 \sqrt{\frac{EI}{\mu \ell^4}}$ rad / s)

3. A compressor blade in a gas turbine can be simulated for vibration analysis by a cantilever of length $\ell$, constant width *b* and depth varying linearly from *d* at the root to zero at the tip. Estimate the first natural frequency of transverse vibration, in the direction of the depth, using Rayleigh's method and assuming the dynamic deflected shape to be the same as the static.

Take Young's Modulus and density of the material as *E* and $\rho$ respectively.

(**Ans**: $\omega_n = 1.581 \frac{d}{\ell^2} \sqrt{\frac{E}{\rho}}$ rad / s $\equiv$ $0.2516 \frac{d}{\ell^2} \sqrt{\frac{E}{\rho}}$ Hz )

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APPLIED MECHANICS - Plate Vibration

Tutorial Sheet 4

1. A rectangular steel, simply-supported plate has sides of length 3 m and 2 m and a thickness of 10 mm. Determine the natural frequencies of the first three modes of transverse vibration and sketch the corresponding nodal lines.

   For steel, $E$ = 210 GN/m$^2$
   $\rho$ = 7850 kg/m$^3$
   $\nu$ = 0.30

   (**Ans:** (i) 8.878 Hz; (ii) 17.07 Hz; (iii) 27.32 Hz; (iv) 30.37 Hz)

2. Part of the deck of a ship may be regarded as a simply-supported rectangular plate of thickness 25 mm and has sides of length 4 m and 10 m. Determine the natural frequencies and mode shapes of the first three modes of vibration.

   What action would you recommend for reducing the response of the deck to an oscillation of frequency 6.5 Hz caused by motion transmitted through the supports from a nearby auxiliary equipment room?

   The plate is made of steel for which $E$ = 210 GPa, $\rho$ = 7850 kg m$^{-3}$ and $\upsilon$ = 0.30.

   (**Ans:** 4.456 Hz, 6.300 Hz, 9.373 Hz)

**Supplementary information:**
**SI_Lumped_parameter_Introductory_Example_sheets_1A&2A**

**These “A” questions were used to provide a confidence building exercise to the later question sheets.**

**VIBRATIONS 2M**

Example Sheet 1A

1. A mass of 20 kg is supported by a spring of stiffness 10 kN/m. What is the undamped natural frequency
   (a) in rad/s,
   (b) in Hz?

   **Ans**: 22.361 rad/s, 3.559 Hz.

2. A dashpot is added to the system in Q1 such that the proportion of critical damping, *c*, is 0.1. What is the dashpot constant, *f* ?

   **Ans**: 89.44 Ns/m.

3. What is the damped natural frequency (i.e. for free vibration) of the system described in Q1 and Q2?

   **Ans**: 22.249 rad/s, 3.541 Hz.

4. What is the period of the undamped and the damped systems?

   **Ans**: 0.281 s, 0.2824 s.

## VIBRATIONS 2M

Example Sheet 2A

1. A cyclic force of 10 N acts on a mass of 10 kg which is supported by a spring of stiffness 10 kN/m. A damper is in parallel with the spring such that $c$ = 0.1.

   What is the undamped natural frequency?

   What is the damped natural frequency?

   What is the resonant frequency, i.e. the frequency at which the maximum amplitude of oscillation is reached? What is this maximum amplitude?

   Sketch the frequency response function, noting the amplitude of vibration at zero frequency, resonance, 0.5 $\omega_n$, 2 $\omega_n$, 5 $\omega_n$, and 10 $\omega_n$.

   **Ans**: 5.033 Hz, 5.0077 Hz, 4.982 Hz, 5 mm.

2. A vibrating system experiences a cyclic force due to a rotating out of balance mass of 2 kg at a radius of 5 mm. What is the magnitude of the exciting force at 1 Hz, 5 Hz and 20 Hz?

   **Ans**: 0.3948 N, 9.87 N, 157.9 N.

3. In a vibrating system, a spring of stiffness 25 kN/m is in parallel with a viscous damper which has a constant of 100 Ns/m. The system is vibrating at 25 Hz. What is the force transmitted to the surroundings

   (a) in the spring,
   (b) in the dashpot,
   (c) in total,

   If the amplitude of vibration is 10 mm?

   **Ans**: 250 N, 157 N, 295.25 N.

**Supplementary information: SI_Lumped_parameter_Examples_sheets**

Department of Mechanical Engineering

VIBRATIONS 2M

Sheet 1 - Free Vibration

1. In a certain ballistics test, a bullet of mass 0.05 kg is fired horizontally into a sand box of mass 100 kg. The sand box can move horizontally on rails and is restrained by a spring and dashpot such that the undamped natural frequency is 1 Hz and the damping is 10% of critical.

   Records from one test indicated that the sand box had a maximum displacement of 0.3 m from the equilibrium position. What was the velocity of the bullet just before it struck the box?

   (**Ans**: 4373 m/s)

2. A door is closed under the control of a spring and dashpot. The spring gives a torque of 13.5 Nm when the door is shut and has a stiffness of 50 Nm rad-1. The dashpot gives a damping torque of 100 Nm s rad-1. The moment of inertia of the door about its hinges is 90 kg m2.

   If the door is opened to 90°, show that it takes a little under 3.6 seconds for it to close.

3. Supplies are parachuted from an aircraft in an experimental container which consists essentially of a spring of stiffness 20 kN m-1 in parallel with a viscous damper of which the dashpot constant is 4472 N s m-1. The free ends of the spring and dashpot are connected together to a light plate which is so designed that it is held firmly by the ground after the first contact. The spring/dashpot combination is arranged to give a total free travel of 1 m.

   The specification requires that the spring/dashpot combination should not "bottom" (so as to avoid excessive shocks) and that undue oscillation should not occur after impact. If the total mass of the container and supplies is 1000 kg and the terminal velocity of the parachute descent is 5 m $s^{-1}$, will the specification be met?

   (0.409 m, 0.5095 m are key values in your solution)

Department of Mechanical Engineering

VIBRATIONS 2M

Sheet 1

1. The rotor of a galvanometer is restrained torsionally by a spiral hairspring and a viscous torsional damper. When the rotor is in its normal position of equilibrium, the pointer indicates zero on the scale of the instrument. The scale is graduated in equal divisions and the pointer reads 100 divisions when the rotor has turned through 40º, the torque required to hold the rotor in this position being 1.25 $\mu$N m. When the rotor is released from rest in this position, the pointer swings to a reading of -10 divisions and then +1 division, the time for each swing being 2 s.

   What is the moment of inertia of the rotor and the undamped natural frequency of the system? If it were required to make the motion aperiodic, by how much must the damping be increased?

   **Ans:** $0.472 \times 10^{-6}$ kg m$^2$; 0.31 Hz (1.947 rad/s); 1.69 times.

2. A door is closed under the control of a spring and dashpot. The spring gives a torque of 13.5 Nm when the door is shut and has a stiffness of 50 Nm rad$^{-1}$. The dashpot gives a damping torque of 100 Nm s rad$^{-1}$. The moment of inertia of the door about its hinges is 90 kg m$^2$.

   Derive an expression for the motion of the door when it is opened 90 degrees and released from rest, and show that it takes a little under 3.6 seconds to close.

   **Ans:** $e^{-0.5555t}$ (1.8408 cos 0.4969t + 2.0580 sin 0.4969t) (rads)

3. Supplies are parachuted from an aircraft in an experimental container which consists essentially of a spring of stiffness 20 kN m$^{-1}$ in parallel with a viscous damper of which the dashpot constant is 4472 N s m$^{-1}$. The free ends of the spring and dashpot are connected together to a light plate which is so designed that it is held firmly by the ground after the first contact. The spring/dashpot combination is arranged to give a total free travel of 1 m.

   The specification requires that the spring/dashpot combination should not "bottom" (so as to avoid excessive shocks) and that undue oscillation should not occur after impact. If the total mass of the container and supplies is 1000 kg and the terminal velocity of the parachute descent is 5 m s$^{-1}$, will the specification be met?

   (Yes, 0.409 m, 0.5095 m are key values in your solution)

4. A motor car of mass 1000 kg has its weight equally distributed on its four wheels. Before fitting the shock absorbers, the car was found to oscillate vertically with a natural frequency of 0.5 Hz. What is the spring rate for each wheel? Determine the dashpot constant to give the vehicle 40% critical damping.

When travelling at 120 km/h, the car is driven along a road where there is a sudden drop in level of 50 mm. Determine the equation of motion of the ensuing vibration about the new equilibrium position. After one complete oscillation, how far will the car have travelled and what will be the displacement relative to this new equilibrium position?

**Ans:** 2467.4 N/m 628.32 Ns/m

$u = e^{-1.2566t}$ [-50 cos 2.8793t - 21.822 sin 2.8793t] (mm)

72.74 m, -3.2217 mm

Nov-94

Department of Mechanical Engineering

VIBRATIONS 2M

Sheet 2

1. A mass of 100 kg is attached to a rigid support by a spring of stiffness 16,000 N/m and is subjected to a harmonic force of amplitude 30 N at the undamped natural frequency. The damping may be considered to be viscous with a coefficient of 300 N.sec $m^{-1}$. Determine (a) the undamped natural frequency, (b) the damped natural frequency, (c) the resonant frequency, (d) the amplitude of the motion of the mass, (e) the phase of the motion relative to the impressed force, and (f) the force transmitted to the support.

   **Ans**: (a) 2.013 Hz, (b) 1.999 Hz, (c) 1.985 Hz, (d) 7.906 mm,
   (e) 90º, (f) 131 N.

2. A printing press weighs 1.8 tonnes and generates a fundamental disturbing force of 2000 N amplitude at a frequency of 1450 c/min. It is supported on four mountings each carrying an equal proportion of the total load. Each mounting has a spring rate of 1.3 x $10^6$ N/m and gives viscous damping with a damping ratio of 0.1 times critical.

   Determine the amplitude of vertical motion of the press and the amplitude of the force transmitted to the foundations.

   **Ans:** 5.48 x $10^{-5}$ m, 328 N.

3. A machine of mass 550 kg is supported on rubber mountings which provide a force proportional to the displacement of 210 kN $m^{-1}$ together with a viscous damping force. The machine gives an exciting force of the form $R\,\omega^2 \cos \omega t$, where $R$ is a constant. At very high rotational speeds, the measured amplitude of vibration was 0.25 mm while the maximum amplitude recorded as the speed was slowly increased from zero was 2 mm. Determine the value of $R$ and the damping ratio $c$.

   **Ans:** 0.1375 kg m, 0.0626.

4. A car plus driver travels over an inferior road surface which has an amplitude of 10 mm and a wavelength of 20 m. The car has a mass of 800 kg and each of the four springs has a stiffness of 10 000 N/m. The net viscous damping ratio for the car and suspension gives $c$ = 0.25. Over what range of speeds will the steady state amplitude exceed 20 mm? What speed needs to be exceeded if the steady state amplitude is to be reduced to below 2 mm?

   **Ans:** 18 to 24.4 m/s, 72.3 m/s.

5. It has been decided to install a precision optical machine in a building near a busy railway line. In the only room available for the machine, it was found that significant vibration levels were caused by passing trains. Analysis of the signals over a period showed that two frequencies predominated, 24 Hz and 11 Hz. At 24 Hz, the peak acceleration was 0.6 g, while at 11 Hz the peak acceleration was 0.4 g.

   The maximum vibration levels permitted for the optical machine were 0.2 mm peak displacement up to 15 Hz, and 18.85 mm/s peak velocity from 15 to 100 Hz.

   An engineering consultant suggested that the machine be mounted on springs. The system chosen provided an undamped natural frequency of 5 Hz and 20% of critical damping. Would this solution be satisfactory? If not, what would you suggest?

   **Ans**: No, it is not satisfactory and the undamped natural frequency needs to be reduced to about 4 Hz.

Department of Mechanical Engineering

VIBRATIONS 2M

Sheet 3

1.

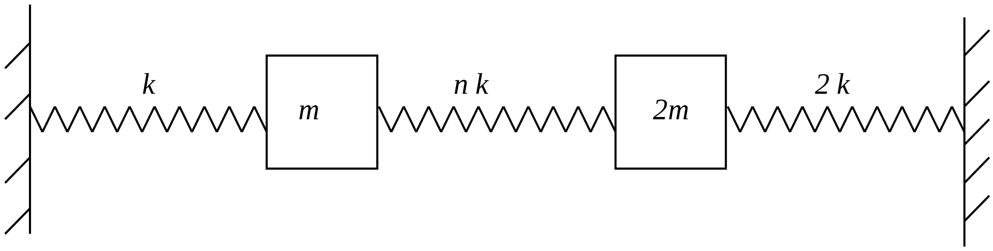



Fig. Q1

Figure.Q1 shows a system which is constrained to vibrate axially. Show that one of the natural frequencies and both of the mode shapes are independent of the constant *n*.

**Ans**: $\omega_1^2 = k/m$ (rad/s)$^2$ ; $\omega_2^2 = k(2 + 3n)/2m$ (rad/s)$^2$
$(u_1/u_2)_1 = 1$ ; $(u_1/u_2)_2 = -2$

2. Determine the natural frequencies and the corresponding mode shapes of the fixed-fixed, two degrees of freedom system sketched in Fig. Q2.

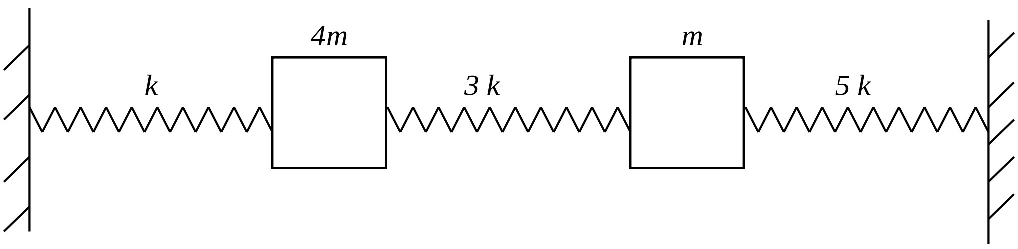



Fig. Q2

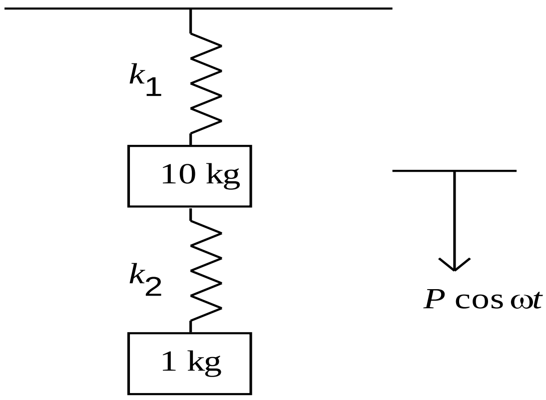
**Ans**: $f_1 = 0.1325\sqrt{k/m}$ Hz ; $f_2 = 0.4589\sqrt{k/m}$ (Hz)
$(u_1/u_2)_1 = 0.4105$ ; $(u_1/u_2)_2 = -9.744$

3. A system vibrating axially can be represented by two discrete masses 2 kg and 4 kg, attached to separate abutments by springs of stiffness $1.10^6$ N/m and $3.10^6$ N/m respectively, and connected to each other by a spring of stiffness $2.10^6$ N/m. If the deflection of the connecting spring is 0.5 mm when the system is vibrating in its second mode, what is the amplitude of displacement of the 2 kg mass?

**Ans**: 3. 0.31 mm

4. A machine may be modelled as the two degree of freedom system sketched in Fig. Q4. The excitation is equivalent to a force $P \cos \omega t$ (where $P$ = 50N) acting on the 10 kg mass as shown. Determine the resonant frequencies $\omega_1$ and $\omega_2$ and the amplitude of motion of the 10 kg mass at a frequency of $(\omega_1 + \omega_2)/2$. What is the value of the detuned frequency?

Fig. Q4




$k_1 = k_2 = 10$ kN m$^{-1}$

**Ans**: $\omega_1 = 30.16$ rad/s ; $\omega_2 = 105.35$ rad/s

1.1295 mm ; $\omega_{det} = 100$ rad/s

8 March '95

**Supplementary information: SI_2nd_year_vibrations_lab_2DoF_sheets**

Second Year Vibrations Laboratory

**One and Two Degree of Freedom Systems**

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**AIMS**

To allow you to explore and investigate the vibration characteristics of simple one and two degree of freedom (DoF) systems. Also, to allow you to become familiar with standard vibration measurement equipment.

The lab will take approximately 2 $^1/_2$ hours to complete.

**REPORTING REQUIREMENTS**

(1) Details of the measurements taken and calculations made should be included in your lab book. This will need to be marked by the lab demonstrator before you leave.
(2) Your report should include the following in the discussion;

For the l DoF system, discuss the agreement between the experimental results and the theoretical prediction. Is the experimental system a simple spring-mass-damper system?

In the light of the results of the 2 DoF system, discuss how a second mass and spring can be added to a one degree of freedom system to reduce vibration at a given frequency (see Appendix 2 fordetails of the theory).

**OVERVIEW**

Although it may be possible to analyse the complete dynamic response of a system, this often leads to complex analysis and the production of large amounts of data. Even if the full dynamic response is required, a first step in any vibrational analysis is to attempt to model the system as either a one or a two degree of freedom system. In this way, much physical insight can be gained and the results act as a useful check on the full results produced later.

When modelling a real system, simplifying assumptions are made. For example, a distributed mass maybe considered as a lumped mass, the effect of damping may be ignored, a non-linear spring may be assumed to be linear over a limited range of motion, and the possible directions of motion restricted. As with any modelling, there is a compromise between simplicity and accuracy.

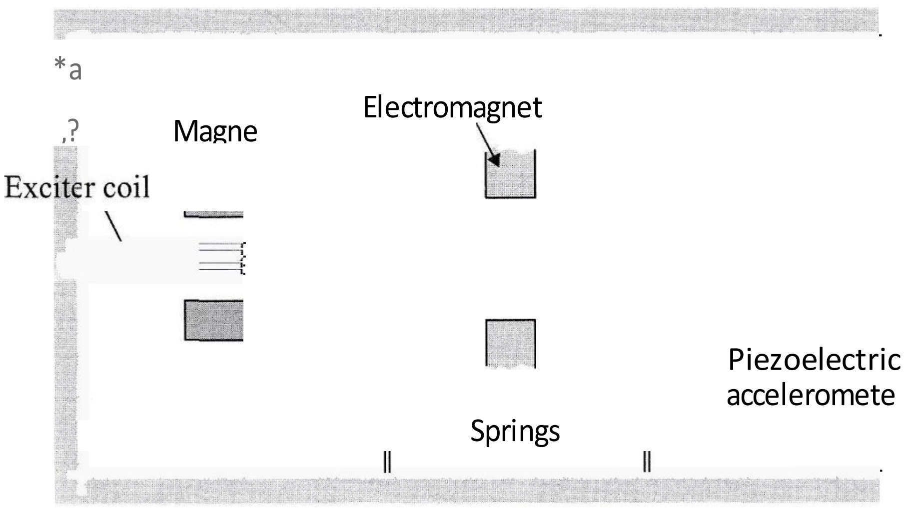
**PART 1 - ONE DEGREE OF FREEDOM SYSTEM**

***Introduction***

Figure 1(a) shows the apparatus used in this experiment. This a close approximation to a one degree of freedom system which is shown schematically in Figure 1(b) and consists of a Mass (acceleration proportional to net external force), a *Spring* (force proportional to displacement), and a *Damper* (force proportional to velocity).

At low frequency (low acceleration), when the force required to accelerate the mass is low, the spring stiffness dominates the motion ('stiffness controlled'). At high frequency (high acceleration), the mass dominates ('mass controlled'). Between these 'high' and 'low' frequency regions, the mass and stiffness cancel each other out when they are opposite in phases and equal in magnitude. At this point, the excitation force has only to overcome the 'damping' or viscosity term. If this is small, the motion of the system becomes very large, as the applied force builds up resonant vibrational energy. Most resonant behaviour will have some mechanism of this kind underlying it; one of the skills of the vibration engineer is that of identifying and determining the effective 'size' of the three fundamental components for any practical case. You need to decide what is acting as a stiffness, what as a mass and what as a damping term.

(a)




(b)

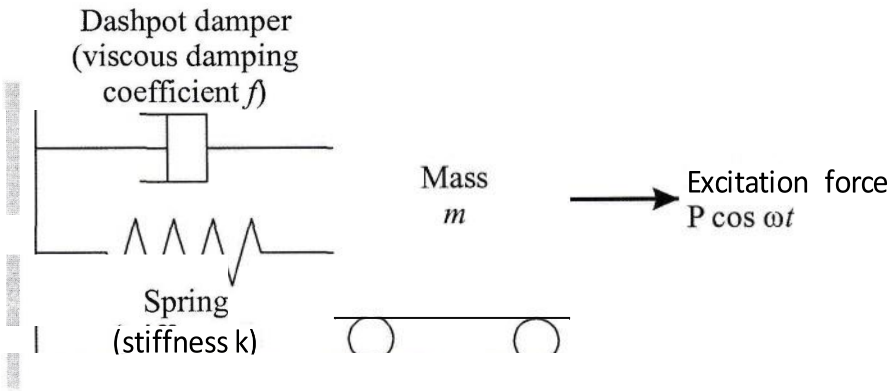



**Figure 1** **(a) Experimental apparatus used for one degree of freedom system and (b) schematic diagram of the components of a one degree of freedom model.**

***Apparatus***

The system has the following features,

- The mass of the moving system consists not only of the block of aluminium and its copper insert, but also of the permanent magnet which is used to interact with the a.c. field produced by current flowing in the coil, thus giving a cyclic force.
- The spring is, in fact, four equal beam springs (one for each 'leg') - these all act together as one but may only be linear for small displacements.
- Induced eddy currents are used to provide damping (the copper insert is there because it has lower resistance than aluminium, so the eddy currents are higher, giving a greater damping force.

- The applied force is independent of the position of the mass only so long as the coil fills the gap of the magnet. This limits the validity of the simple model to small amplitudes.

In addition, the following instrumentation is provided.

- A sine wave generator connected to a power amplifier, the output of which is fed to the coil to induce the electromagnetic excitation force. ***Note that the coil can only take currents up to 0.1 amp, so take care to monitor this***.
- A piezoelectric accelerometer mounted on a small magnet (to be placed on the mass) and connected to a preamplifier. The calibration will have been set for m/s2. The output is connected in turn to an oscilloscope and a voltmeter so that the vibration response of the system can be measured.
- An ammeter to measure the d.c. current to the damping coil.

### *Experimental Procedure (One Degree of Freedom)*

(1) Familiarise yourself with the equipment (the lab demonstrator will help you.

(2) Perform a coarse frequency sweep with zero damper current to find the approximate resonance frequency. **Note**: the fastest way to find the resonance frequency is to make a rough plot as you go along.

(3) Carry out a finer frequency sweep in the region of the resonance frequency.

(4) Use the half power point method to determine the damping ratio of the system (see Appendix).

(5) Vary the damper current and observe the effects on the resonance amplitude, $\hat{u}$, and frequency, $F_{res}$ PlOt graphs of $\hat{u}$ and $F_{res}$ against the damper current up the maximum value advised by the demonstrator. Comment on the results.

## PART 2 - TW0 DEGREES OF FREEDOM SYSTEM

### *Introduction*

If there is more than one independent mass in a system, it may begin to exhibit more complex behaviour. There are several ways in which even a small number of extra components can be interconnected. We have constructed one of the simplest, which in its practical form is shown in Figure 2(a). Such a system can be represented by the block diagram shown in Figure 2(b) and asimple mathematical model constructed.

### *Apparatus*

In addition to the apparatus described in Part 1, a second mass is now added to the one degree of freedom system. The vibration response of this mass is monitored with a second accelerometer which should be connected via a preamp to channel 2 of the oscilloscope.

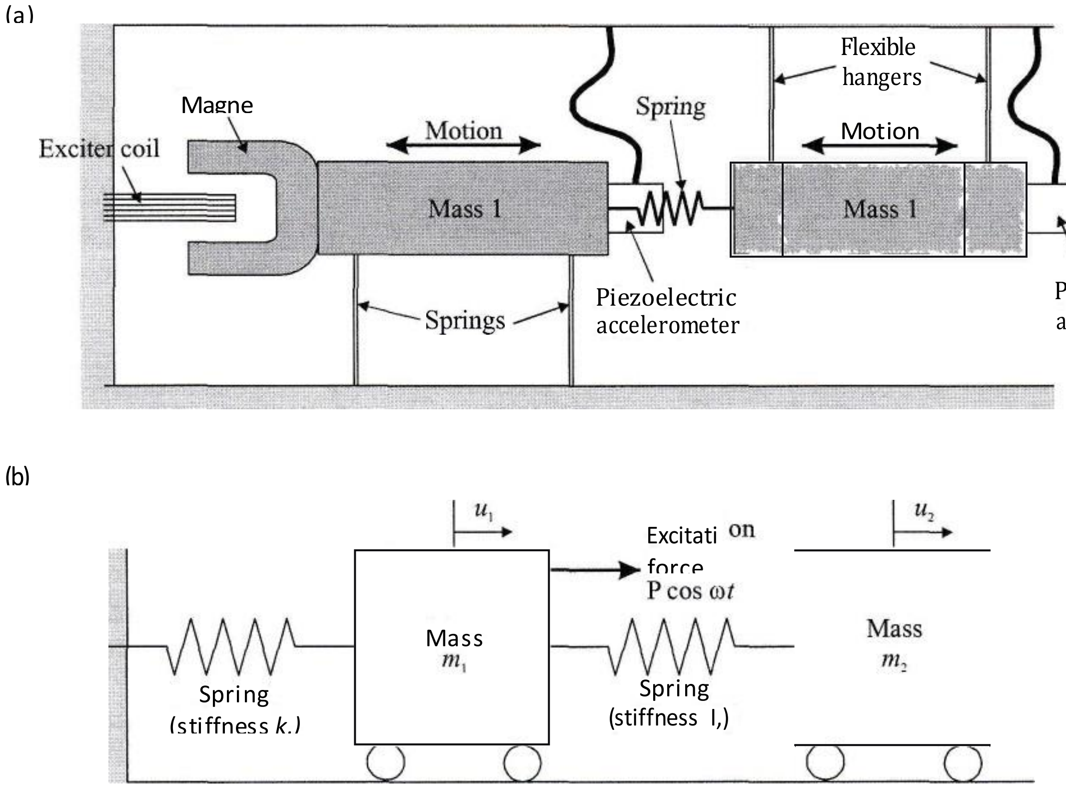



**Figure 2** **(a) Experimental apparatus used for two degrees of freedom system and (b) schematic diagram of the components of a two degrees of freedom model.**

***Experimental Procedure (2 Degrees of Freedom)***

(1) Check the frequency response function for the single mass.

(2) Connect the second mass and the additional accelerometer.

(3) Perform the same frequency sweeps as for the 1 DoF system. For this system you will see two resonance frequencies with a detuned zone between them.

(4) Focus finer sweeps on the two resonant frequencies and the detuned zone.

***USE OF THE SPREADSHEET***

(1) You will be given the two masses.

(2) Input the data for the 1 DoF system and the 2 DoF system, being careful to place the correct amplitude in the correct column for the 2 DoF. **Note**: input the data only in the blue cells. **Do not** change the yellow cells.

(3) For the 1 DoF system, adjust the theoretical values for the spring constants, damping constants and excitation force to fit the theoretical curve to the experimental data. Note that the theoretical values you input for the I DoF system will be transported automatically across for use in the 2 DoF models.

## Appendix

### THE DETUNED FREQUENCY OF A TWO DEGREES OF FREEDOM SYSTEM

The amplitude-frequency characteristic of the two degrees of freedom system has two resonances (amplitude peaks). In addition there is a frequency at which an amplitude minimum occurs called the detuned frequency. In Figure 2(b), the mass ml is being forced to oscillate at an angular frequency, cv, by a cyclic force, Pcoscvt. Because, m1 and m2 are connected by the spring k2, both u1 and u2 will vibrate in some collective manner. The equations of motion of the two masses are:

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = P\cos\omega t$$

$$m_2\ddot{u}_2 - k_2u_1 + k_2u_2 = 0$$

and for *m2.* By substituting, *u 1* = *U1 coscvt* and *u 2* = *U 2 coscvt* it can be shown that:

U = P(kz - m2cv²)

t m1 m2 0J -(m2k1 +m2k2 + m1 k )cv² +kl2

Resonances occur when the denominator is equal to zero. Also note that when cv² = *k2* / *m2* , *U1* is equal to zero and the frequency at which this occurs in this two degrees of freedom system is called the *detuned frequency.*

The detuned frequency concept is commonly used in vibration detuners or absorbers (an absorber is a damped detuner). Consider a one degree of freedom system with resonance at:

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}$$

With the addition of an extra spring and mass, this system can be converted into a two degrees of freedom system. The above equation for *U1* will now determine the amplitude of vibration of the first mass and, when the numerator equals zero, there will be no vibration. The situation for no vibration is when:

$$\omega_{\text{det}} = \sqrt{\frac{k_2}{m_2}}$$

By choosing the values of the new spring and mass, then this zero vibration can occur at any troublesome frequency, and even at the resonance frequency of the original system, thereby causing the mass *m* 1 to be at rest.

**Supplementary information: SI_3rd-year-vibrations-lab-plates_sheets**

Third Year Vibrations Laboratory

## The Vibration Characteristics of Rectangular Plates

### AIMS

To demonstrate the vibration characteristics (natural frequencies and mode shapes) of free-free rectangular plates. Also to allow you to investigate the effect of plate geometry on these vibration characteristics.

The lab will take approximately 2½ hours to complete.

### REPORTING REQUIREMENTS

1) In the laboratory you should detail the measurements taken and the calculations made. This will be checked at the end of the class.
2) You should write a two side (about 600 words) executive summary of the lab. This should include a brief description of the aims and objectives. It should also include brief details of the key experimental findings and a discussion of their meaning. The experimental data, graphs and calculations should be attached as an appendix.

### OVERVIEW

A vibrating plate is a two-dimensional system. Algebraic solutions of the equation of motion can only be determined for simple shapes, such as rectangles and circles, and with certain types of boundary conditions. For a rectangular plate in the *x-y* plane, the equation of motion is:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{12\rho\left(1-v^2\right)}{Eh^2}\frac{\partial^2 w}{\partial t^2} = 0$$

where *w* is the deflection in the *z* direction, $\rho$ is the density, *E* is Young's modulus, *v* is Poisson's ratio and *t* is time.

This differential equation can only be solved explicitly for certain conditions. In particular, if the plate sides are simply supported (hinged), then a solution of the form:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(T\cos\omega t + U\sin\omega t\right)$$

is possible where *a* and *b* are the side lengths in the *x* and *y* directions, *m* and *n* define the number of half waves in the *x* and *y* directions, $\omega$ is the circular frequency ( $2\pi$ times frequency, *F)* and *T* and *U* are arbitrary constants.

It is difficult to model physically the simply supported edge condition. Clamped edges are also difficult to realise. However, free edges can be achieved easily, the only disadvantage being that it is not possible to obtain an explicit algebraic function for *w*(*x,y*). However, a numerical solution can be obtained to determine the frequencies, mode shapes and nodal positions:

perhaps the most popular method of solution today is to use a finite element package, or some other PC-based solution.

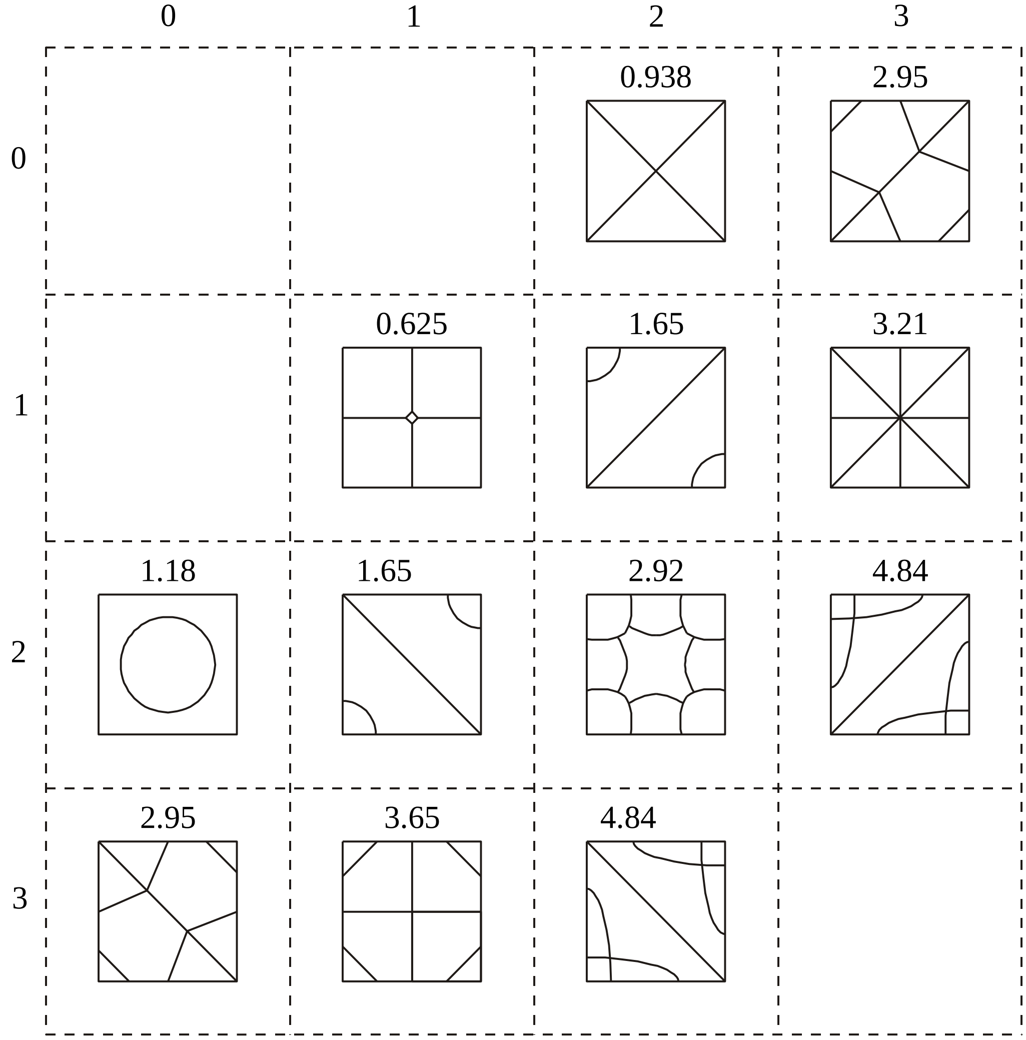



**Figure 1** **Mode shapes of a free-free square plate (the values of B are shown for each mode).**

A series of nodal patterns for a free-free square plate as predicted by a finite element model are shown in Figure 1. The constant B is used to define the natural frequency *F* for each mode using the equation,

$$F = B\frac{cd}{\ell^2\sqrt{1-v^2}}$$

where, *c* is the velocity of extensional waves $\left(=\sqrt{E/\rho}\right)$, *d* is the plate thickness, $\ell$ is the side length, and $v$ is Poisson's ratio.

For non-square (rectangular) plates, the natural frequencies are related to the side lengths *a* and *b.* As for the square plates, the frequencies have to be predicted by numerical techniques as there is no explicit relationship for *w*(*x, y*) which fits the differential equation of motion and the boundary conditions (except for simply supported edges).

When *a>>b,* the plate approximates to a beam and the solution is as in the Vibrating Beams experiment. Between the beam and square plate lies an interesting transition zone, in which we need to answer the question 'What is a beam and what is a plate?'

## Apparatus

The apparatus consists of three main systems, the shaker excitation system, the microphone – vibration measurement system and a system for stroboscopic mode shape determination. The main components of these systems are,

Shaker excitation system

- Sine wave generator (low voltage)
- Power amplifier (amplifies the sine wave and supplies alternating current to the)
- Electromagnetic shaker (which drives the plate)

Microphone vibration measurement systems

- Microphone ('measures' the vibration)
- Charge amplifier (amplifies the microphone output)
- Oscilloscope (displays the results)

Stroboscopic mode shape determination

- Strobe light
- Strobe exciter (triggered from the sine wave generator to ensure synchronisation)

## EXPERIMENTAL PROCEDURE

### *Part 1 - Square Plate*

This experiment will demonstrate mode shapes, nodal positions, and resonance frequencies in square plates.

- Determine the first six natural frequencies and the corresponding nodal patterns. Note that the stroboscope can be controlled to flash at the excitation frequency, thus freezing the mode shape. A more interesting feature is to flash at 1~3 Hz below the excitation frequency which causes the plate to move in slow motion. Sketch the mode shapes.
- Compare these with the theoretical predictions. For aluminium material properties, use $E$ = 70 GPa, $\rho$ = 2750 kg/m$^3$, and $v$ = 0.35.

Note that some pairs of modes occur at the same frequency but with the nodal patterns rotated through 90°. The observed mode will be some combination of these two modes. By holding the plate at a node for one of the modes (e.g. a corner), it is possible to suppress the other mode by damping since this corner would otherwise be an antinode.

### *Part 2 - The Transition Series*

The transition series consists of a set of aluminium plates varying from an almost square plate to one which is effectively a beam.

- For the beam, determine the first and second bending modes of vibration. Refer to the beam vibration experiment if you are unsure what this means.
- Take the next narrowest plate and determine the same two bending modes. These should be at very similar frequencies to the original beam.

- Repeat this series of tests up to the almost square plate. Note the resonant frequencies and the nodal patterns.
- Now, start with the nearly square plate and determine its lowest resonant frequency. This should give a nodal pattern similar to that for the 1,1 mode of the square plate shown in Figure 1.
- Follow the transition series in reverse order compared with the first part of the experiment. Excite the 1,1 mode in each of the rectangular plates: you may have trouble with the beam! Determine the frequency and nodal pattern for the 1,1 mode for each plate.
- Plot the three frequencies you have calculated against the ratio $b/\ell$, where *b* is the beam width and $\ell$ is its length.

**EXPERIMENTAL HINTS**

*Position of shaker (driver, exciter)* - you cannot excite a system at a node. On the other hand, if you position the driver at an antinode, it may not be able to follow the motion owing to its internal constraints. Judicious positioning of the shaker will achieve the best results. You may move either the plate or the shaker (the plate is easier).

*Position of detector* - in both experiments, the detector is a small microphone: this is non-contacting and is easily moved to determine the nodes and the amplitude of motion. Place the microphone over an antinode.

*Position of supports* - the supports are made of soft foam so as to minimise their effect on the plate in terms of stiffness and mass. Within reason, try and move the supports to the nodal positions: this is not easy, especially with the plate. The best rule is to avoid having the supports at antinodes.

*Determination of nodes* - dry sand is to be used for this purpose. The sand is agitated and will move either to a position where the acceleration is less than g, i.e. towards a nodal line, or it will fall off the edge. But try also a lightly-held pencil, or your finger nail: these are also sensitive to motion.

*Frequency Control* - the electronic oscillator can be swept manually over a wide frequency range. Start at about 50 Hz and work upwards, watching the oscilloscope screen for resonance.

Note: It is sometimes possible to excite resonance when the drive frequency is only a half of the resonant frequency. This is due to non-linearity in the load-deflection characteristics of the foam block used to couple the exciter to the beam. The difference in frequency of the two signals is obvious on the oscilloscope screen. Avoid this problem by simply returning to twice the original frequency when you should find resonance properly.